Some Basic Definitions

Trials & Events : If an experiment is repeated under essentially the same conditions and it result in any one of the several possible outcomes, then the experiment is called a **trial** and the possible outcomes are known as **events**. Eg : Tossing of a coin is **trial** and turning up of head/tail is and **event**.

Exhaustive Events: The outcomes of a random experiment is called **exhaustive events** if it covers all the possible outcomes of the experiment. Eg : In rolling of a die, the outcomes 1,2,3,4,5,6 are exhaustive events.

Favourable Events: The events which entail the required happening are called favourable events. Eg : In throwing of two dice the number of favourable cases of getting sum 7 is **6** viz. (1,6), (6,1), (2,5), (5,2), (3,4), (4,3).

Mutually Exclusive Events: Two or more events are said to be mutually exclusive if occurrence of one of them excludes the occurrence of the other. Eg : While tossing a coin we either get a head or a tail but not both.

Independent Events: Two or more events are said to be independent events if happening or non-happening of one doesn't depend on the happening or non-happening of the other. Eg : Two coins tossed at the same time, the outcome of one is independent of the outcome of the other.

Equally likely events: Two events are said to be equally likely if there is no reason to expect anyone with preference to other. Eg : Head and tail are equally likely to come.

Sample Space: The set of all possible outcomes of an experiment is called sample space. It is denotes by **S**.

Classical Definition of Probability

Let **S** be the sample space and **E** be an event. Then probability of occurrence or happening of **E** is denoted by **P(E)** and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ outcomes}$$

Axioms of Probability

Let **S** be the sample space and **E** be an event. Then

- $0 \leq P(E) \leq 1$
- $P(S)=1$
- If $E_1, E_2, ..., E_n$ be n mutually exclusive events then

$$P(E_1 \cup E_2 \cup ... \cup E_n)=P(E_1)+P(E_2)+....+P(E_n)$$

Notes

- $P(\phi)=0$
- If E be an event then Probability of non-happening of E is denoted by $P(\bar{E})$ or $P(E')$ and is given by $P(\bar{E})=1-P(E)$.
- $P(\bar{A}\cap B)=P(B)-P(A\cap B)$
- $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

  If A and B are mutually exclusive then $A\cap B=\phi \Rightarrow P(A\cap B)=0$

  Then $P(A\cup B)=P(A)+P(B)$

- $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

  If A, B and C are mutually exclusive then $P(A\cup B\cup C)=P(A)+P(B)+P(C)$

Conditional Probability

Let A and B be any two events of a random experiment. Then Probability of occurrence of A given that B has already occurred is denoted by P(A/B) and is defined as

$$P\left({}^{A}/_{B}\right) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Properties

Let A and B be any two events of a sample space S then

$$P\left({}^{S}/_{B} \right) = P\left({}^{B}/_{B} \right) = 1$$

$$P\left({}^{(A \cup B)}/_{F} \right) = P\left({}^{A}/_{F} \right) + P\left({}^{B}/_{F} \right) - P\left({}^{(A \cap B)}/_{F} \right)$$

$$P\left({}^{E'}/_{F} \right) = 1 - P\left({}^{E}/_{F} \right)$$

Questions

1. A pair of dice is rolled, find $P\left({}^{A}/_{B} \right)$ if

   A : 3 appears on one die

   B : sum of number appearing is 7

2. A card is drawn from a well shuffled deck of 52 cards and then a second card is drawn, find

   The probability that the first card is a spade and then second card is a club if the first card is not replaced.\

A card is drawn from a pack of well – shuffled palying cards. What is the probability that it is either a spade or an ace ?

Let A be the event of drawing a sapde

B be the event of drawing an ace

We know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad ....(i)$$

Now,

$$n(A) = 13 \qquad n(B) = 4 \qquad n(A \cap B) = 1 \qquad n(S) = 52$$

$$P(A) = \frac{13}{52} \qquad P(B) = \frac{4}{52} \qquad P(A \cap B) = \frac{1}{52}$$

Substituting the above values in (i) we get

$$P(A \cup B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Multiplicative Law of Probability

The probability of simultaneous occurrence of two events is equal to the probability of one multiplied by the conditional probability of the other *i.e.* if **A** and **B** be two events then probability of simultaneous occurrence of both is given by

$$P(A \cap B) = P(A) \, . P\left({}^{B}/_{A} \right)$$

Note

$P(A \cap B)$ *is also written as* $P(AB)$

If A and B are independent events then $P(A \cap B) = P(A)\, P(B)$

A problem in mechanics is given to three students A, B, C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that the problem will be solved?

$P(A)=\frac{1}{2}$ $P(B)=\frac{1}{3}$ $P(C)=\frac{1}{4}$

$P(A')=1-\frac{1}{2}=\frac{1}{2}$

$P(B')=1-\frac{1}{3}=\frac{2}{3}$

$P(C')=1-\frac{1}{4}=\frac{3}{4}$

Probability that A, B, C cannot solve the problem $i.e.$ probability that the problem will not be solved $=P(A').P(B').P(C')=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$

Probability that atleast one of them will solve problem $i.e.$ probability that the problem will be solved $=1-\frac{1}{4}=\frac{3}{4}$

A can hit a target 4 times in 5 shots, B 3 times in 4 shots, C twice in 3 shots. They fire a volley. What is the probability that atleast two shots hit?

Probability that A hit the target $= P(A) = \frac{4}{5}$

Probability that B hit the target $= P(B) = \frac{3}{4}$

Probability that C hit the target $= P(C) = \frac{2}{3}$

Case 1 : A, B, C all hit the target then

$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C) = \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$

Case 2 : *A, B* hit the target but *C* misses it, then

$$P(A \cap B \cap C') = P(A)\,P(B)\,P(C') = \frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{12}{60}$$

Case 3 : *A, C* hit the target but *B* misses it, then

$$P(A \cap B' \cap C) = P(A)\,P(B')\,P(C) = \frac{4}{5} \times \left(1 - \frac{3}{4}\right) \times \frac{2}{3} = \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} = \frac{8}{60}$$

Case 4 : B, C hit the target but A misses it, then

$$P(A' \cap B \cap C) = P(A')\,P(B)\,P(C) = \left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{6}{60}$$

$$\textit{So the required probability} = \frac{24}{60} + \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{50}{60} = \frac{5}{6}$$

A and B throw alternatively with a single die. A having the first throw. The person who first throws an *ace* is to win. What are their respective chances of winning ?

$$Probability\ of\ \ getting\ an\ ace\ in\ throwing\ of\ \ a\ die\ = \frac{1}{6}$$

$$Probability\ of\ \ not\ getting\ an\ ace\ in\ throwing\ of\ \ a\ die\ = 1 - \frac{1}{6} = \frac{5}{6}$$

$$Case\ I : If\ \ A\ is\ to\ win\ then\ he/she\ must\ throw\ an\ ace\ in\ 1^{st}\ or\ 3^{rd}\ or\ 5^{th} \ldots\ throws$$

$$Probability\ that\ A\ gets\ an\ ace\ in\ 1^{st}\ throw\ = \frac{1}{6}$$

$$Probability\ that\ A\ gets\ an\ ace\ in\ 3^{rd}\ throw\ = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$Probability\ that\ A\ gets\ an\ ace\ in\ 5^{th}\ throw\ = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$Then\ Probability\ that\ A\ will\ win\ = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \ldots. = \frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

$Case\ 2:\ If\ \ B\ is\ to\ win\ then\ he/she\ must\ throw\ an\ ace\ in\ 2^{nd}\ or\ 4^{th}\ or\ 6^{th}\ \ \ldots\ \ throws$

$$Probability\ that\ B\ gets\ an\ ace\ in\ 2^{nd}\ throw\ = \frac{5}{6} \times \frac{1}{6}$$

$$Probability\ that\ B\ gets\ an\ ace\ in\ 4^{th}\ throw\ = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$Probability\ that\ B\ gets\ an\ ace\ in\ 6^{th}\ throw\ = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$Then\ Probability\ that\ B\ will\ win\ = \left(\frac{5}{6}\right) \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6} + \ldots. = \frac{\frac{5}{6} \times \frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} = \frac{5}{11}$$

$Alternate\ method\ to\ calculate\ B:\ \ P(B) = 1 - P(A)$

A bag contains 10 balls, two of which are red, three blue and five black. Three balls are drawn at random from the bag. What is the probability that

(i) three balls are of different colours

(ii) two balls are of same colour

(iii)the balls are of different colour

$$Total\ number\ of\ balls = 10$$

$$Number\ of\ red\ balls = 2$$

$$Number\ of\ blue\ balls = 3$$

$$Number\ of\ black\ balls = 5$$

$$(i)\ Probability\ that\ all\ the\ three\ balls\ are\ of\ different\ colours = \frac{{}^{2}c_{1} \times {}^{3}c_{1} \times {}^{5}c_{1}}{{}^{10}c_{3}}$$

$$(ii)\ Case\ 1: If\ the\ same\ balls\ are\ white\ balls$$

$$2\ balls\ white,\ 1\ ball\ blue = \frac{{}^{2}c_{2} \times {}^{3}c_{1}}{{}^{10}c_{3}}$$

$$2\ balls\ white,\ 1\ ball\ black = \frac{{}^{2}c_{2} \times {}^{5}c_{1}}{{}^{10}c_{3}}$$

$$Probability\ that\ 2\ balls\ white = \frac{{}^{2}c_{2} \times {}^{3}c_{1}}{{}^{10}c_{3}} + \frac{{}^{2}c_{2} \times {}^{5}c_{1}}{{}^{10}c_{3}}$$

Case 2 : If the same balls are blue balls

$$2\ balls\ blue\ ,\ 1\ ball\ \ white\ = \frac{{}^{3}c_{2} \times {}^{2}c_{1}}{{}^{10}c_{3}}$$

$$2\ balls\ blue\ ,\ 1\ ball\ \ black\ = \frac{{}^{3}c_{2} \times {}^{5}c_{1}}{{}^{10}c_{3}}$$

$$Probability\ that\ 2\ balls\ blue\ = \frac{{}^{3}c_{2} \times {}^{2}c_{1}}{{}^{10}c_{3}} + \frac{{}^{3}c_{2} \times {}^{5}c_{1}}{{}^{10}c_{3}}$$

Case 3 : If the same balls are black balls

$$2\ balls\ black\ ,\ 1\ ball\ \ white\ = \frac{{}^{5}c_{2} \times {}^{2}c_{1}}{{}^{10}c_{3}}$$

$$2\ balls\ black\ ,\ 1\ ball\ \ blue\ = \frac{{}^{5}c_{2} \times {}^{3}c_{1}}{{}^{10}c_{3}}$$

$$Probability\ that\ 2\ balls\ black\ = \frac{{}^{5}c_{2} \times {}^{2}c_{1}}{{}^{10}c_{3}} + \frac{{}^{5}c_{2} \times {}^{3}c_{1}}{{}^{10}c_{3}}$$

$$Required\ Probablity = \left(\frac{3}{^{10}C_3} + \frac{5}{^{10}C_3}\right) + \left(\frac{6}{^{10}C_3} + \frac{15}{^{10}C_3}\right) + \left(\frac{20}{^{10}C_3} + \frac{30}{^{10}C_3}\right) = \frac{79}{120}$$

If A and B are events such that $P(A)=0.3,\ P(B)=p,\ P(A\cup B)=0.6$ then

(i) Find p so the A and B are independent events

(ii) For what value of p, A and B are mutually exclusive

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\Rightarrow P(A\cap B)=P(A)+P(B)-P(A\cup B)=0.3+p-0.6$$

$$\Rightarrow P(A\cap B)=p-0.3$$

(i) *If A and B are independent then* $P(A\cap B)=P(A)P(B)$

$$\Rightarrow p-0.3=0.3p$$

$$\Rightarrow 0.7p=0.3$$

$$\Rightarrow p=\frac{3}{7}$$

(ii) *If A and B are mutually excluisive then* $P(A\cap B)=0$

$$\Rightarrow P(A\cap B)=p-0.3$$

$$\Rightarrow 0=p-0.3$$

$$\Rightarrow p=0.3$$

Q. An integer is chosen at random from first 200 positive integers. What is the probability that the integer chosen is divisible by 6 or 8 ?

Q. *A, B* and *C* in order, toss a coin. The first one to throw a head winds. If *A* starts first find their respective probability of winning.

Q. Find the chance of throwing 5 or 6 at least once in four throws of die.

Q. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled out of a random from one of the two purse, what is the probability that it is silver coins ?

BAYES' THEOREM

If $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events with $P(E) \neq 0,\ (i=1, 2, \ldots, n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P\left(E_i / A\right) = \frac{P\left(E_i\right) P\left(A / E_i\right)}{\sum_{i=1}^{n} P\left(E_i\right) P\left(A / E_i\right)}$$

A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

$$Let,\ E_1 = the\ ball\ is\ drawn\ from\ bag\ X$$

$$E_2 = the\ ball\ is\ drawn\ from\ bag\ Y$$

$$A = the\ ball\ is\ red$$

$$By\ Bayes'\ Theorem,$$

$$P\left(E_2/A\right) = \frac{P\left(E_2\right)P\left(A/E_2\right)}{P\left(E_1\right)P\left(A/E_1\right) + P\left(E_2\right)P\left(A/E_2\right)}$$

$$P(E_1) = Probability\ that\ the\ ball\ is\ drawn\ from\ bag\ X = \frac{1}{2}$$

$$P(E_2) = Probability\ that\ the\ ball\ is\ drawn\ from\ bag\ Y = \frac{1}{2}$$

$$P(A/E_1) = Probability\ that\ a\ red\ ball\ drawn\ from\ bag\ X = \frac{3}{5}$$

$$P(A/E_2) = Probability\ that\ a\ red\ ball\ drawn\ from\ bag\ Y = \frac{5}{9}$$

$$P(E_2/A) = \frac{\frac{1}{2} \times \frac{5}{9}}{\left(\frac{1}{2} \times \frac{3}{5}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)} = \frac{25}{52}$$

An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets an accident. What is the probability that he is a scooter driver?

$$Let\ E_1 = insured\ \ person\ is\ scooter\ driver$$

$$E_2 = insured\ \ person\ is\ car\ driver$$

$$E_3 = insured\ \ person\ is\ truck\ driver$$

$$A = insured\ \ person\ meets\ accident$$

$$By\ Bayes'\ Theorem,$$

$$P\left(E_1/A\right) = \frac{P(E_1)\ P\left(A/E_1\right)}{P(E_1)\ P\left(A/E_1\right) + P(E_2)\ P\left(A/E_2\right) + P(E_3)\ P\left(A/E_3\right)}$$

$$P(E_1) = Probabilty\ that\ the\ insured\ \ person\ is\ scooter\ driver = \frac{2000}{12000} = \frac{1}{6}$$

$$P(E_2) = Probabilty\ that\ the\ insured\ \ person\ is\ car\ driver = \frac{4000}{12000} = \frac{1}{3}$$

$$P(E_3) = Probabilty\ that\ the\ insured\ \ person\ is\ truck\ driver = \frac{6000}{12000} = \frac{1}{2}$$

$$P\left(A/E_1\right) = Probability\ that\ insured\ \ person\ meets\ accident\ given\ he\ is\ scooter\ driver = 0.01$$

$$P\left(A/E_2\right) = Probability\ that\ insured\ \ person\ meets\ accident\ given\ he\ is\ car\ driver = 0.03$$

$$P\left(A/E_3\right) = Probability\ that\ insured\ \ person\ meets\ accident\ given\ he\ is\ truck\ driver = 0.15$$

$$P\left(E_1/A\right)=\frac{P\left(E_1\right)\,P\left(A/E_1\right)}{P\left(E_1\right)\,P\left(A/E_1\right)+P\left(E_2\right)\,P\left(A/E_2\right)+P\left(E_3\right)\,P\left(A/E_3\right)}$$

$$=\frac{\frac{1}{6}\times 0.01}{\left(\frac{1}{6}\times 0.01\right)+\left(\frac{1}{3}\times 0.03\right)+\left(\frac{1}{2}\times 0.15\right)}=\frac{1}{52}$$

A man is known to speak truth 3 out 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

$$Let \ \ A = the \ man \ reports \ it \ is \ a \ six$$

$$E_1 = six \ occurs$$

$$E_2 = six \ doesn't \ occur$$

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$Now,$$

$$P(E_1) = Probability \ that \ six \ occurs = \frac{1}{6}$$

$$P(E_2) = Probability \ that \ six \ doesn't \ occur = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(A/E_1) = \textit{Probability that the man reprots that it is six given that six occurs}$$

$$= \textit{Probability that the man speaks truth} = \frac{3}{4}$$

$$P(A/E_2) == \textit{Probability that the man reprots that it is six given that six doesn't occur}$$

$$= \textit{Probability that the man doesn't speak the truth} = 1 - \frac{3}{4} = \frac{1}{4}$$

So,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{(\frac{1}{6} \times \frac{3}{4}) + (\frac{5}{6} \times \frac{1}{4})} = \frac{3}{8}$$

Suppose that 5% of men and 0.25% of women have a grey hair. A grey haired person is selected at random. What is the probability of this person being a male ? Assume that there are equal number of males and females.

$$\text{Let } A = \text{a grey haired person is chosen}$$

$$E_1 = \text{a male is chosen}$$

$$E_2 = \text{a female is chosen}$$

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

Now,

$$P(E_1) = \text{Probability that a male is chosen} = \frac{1}{2}$$

$$P(E_2) = \text{Probability that a female is chosen} = \frac{1}{2}$$

$$P\left(A/E_1\right) = Probability\ that\ a\ grey\ haired\ person\ is\ chosen\ when\ it\ known\ that\ the\ person\ is\ male\ = \frac{5}{100} = 0.05$$

$$P\left(A/E_2\right) = Probability\ that\ a\ grey\ haired\ person\ is\ chosen\ when\ it\ known\ that\ the\ person\ isf\ emale = \frac{0.25}{100} = 0.0025$$

$So,$

$$P\left(E_1/A\right) = \frac{P\left(E_1\right)P\left(A/E_1\right)}{P\left(E_1\right)P\left(A/E_1\right) + P\left(E_2\right)P\left(A/E_2\right)}$$

$$= \frac{\frac{1}{2} \times 0.05}{\left(\frac{1}{2} \times 0.05\right) + \left(\frac{1}{2} \times 0.0025\right)} = \frac{20}{21}$$

Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die ?

$$Let\ A = getting\ exactly\ one\ head$$

$$E_1 = getting\ 5\ or\ 6\ in\ a\ \text{single}\ throw\ of\ \ a\ die$$

$$E_2 = getting\ 1,\ 2,\ 3\ or\ 4\ in\ a\ \text{single}\ throw\ of\ \ a\ die$$

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$$By\ Bayes'\ Theorem,$$

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$P\left(A/E_1\right) = Probability\ of\ \ getting\ exactly\ one\ head\ given\ that\ a\ coin\ is\ tossed\ three\ times$$

$$= {}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1} = \frac{3}{8}$$

$$P\left(A/E_2\right) = Probability\ of\ \ getting\ exactly\ one\ head\ given\ that\ a\ coin\ is\ tossed\ once$$

$$= \frac{1}{2}$$

$So,$

$$P\left(E_2/A\right) = \frac{P\left(E_2\right)P\left(A/E_2\right)}{P\left(E_1\right)P\left(A/E_1\right) + P\left(E_2\right)P\left(A/E_2\right)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\left(\frac{1}{3}\times\frac{1}{2}\right)+\left(\frac{2}{3}\times\frac{1}{2}\right)} = \frac{8}{11}$$