PARTIAL DIFFERENTIAL EQUATION

Course Title - Mathematics III(A)
Course Code - MA181301A

Syllabus

Formation of Partial Differential equations, Linear partial differential equation of first order, Non-linear partial differential equations of first order, Charpit's method, Method of separation of variables,boundary value problem with reference to the one dimensional heat and wave equation.

Functions of two variables

If x, y, z are any three variables so related that the value of z depends upon the value of x and y then z is called a function of two variables x and y and it is denoted by $z=f(x,y)$.

z is called dependent variable and x and y are called independent variables.

Partial Derivative of First Order

Let $z=f(x,y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary then z becomes a function of x only. The derivative of z *w.r.t.* (with respect to) x, treating y as constant is called partial derivative of y *w.r.t.* x and is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

Similarly, we can define the partial derivative of z *w.r.t.* y and it is denoted by

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y$$

$$\frac{dy}{dx} \rightarrow Differentiation\ of\ \ y\ w.r.t.\ x$$

$$\frac{\partial y}{\partial x} \rightarrow Partial\ Differentiation\ of\ \ y\ w.r.t.\ x$$

Find the first order partial derivative of $z = e^{xy}$

$$z = e^{xy}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^{xy}) = e^{xy}\frac{\partial}{\partial x}(xy) = e^{xy}(y) = ye^{xy}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^{xy}) = e^{xy}\frac{\partial}{\partial y}(xy) = e^{xy}(x) = xe^{xy}$$

If z be a function of two independent variables x and y then

$$\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}=q,\ \frac{\partial^2 z}{\partial x^2}=r,\ \frac{\partial^2 z}{\partial x \partial y}=s,\ \frac{\partial^2 z}{\partial y^2}=t$$

Formation of P.D.E.

Partial Differential Equation can be formed either by the elimination of arbitrary constants or by elimination of arbitrary functions.

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the P.D.E. that arise are of first order.

If the number of arbitrary constants to be eliminated is more than the number of independent variables, the P.D.E. that obtained are of second or higher order.

If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the P.D.E. is in general equal to the arbitrary functions eliminated.

Form partial differential equation from $z = ax + by + a^2 + b^2$

$$z = ax + by + a^2 + b^2 \quad \ldots (i)$$

Differentiating (i) partially w.r.t. x we get

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(ax + by + a^2 + b^2\right) = a$$

$$\Rightarrow p = a \quad \ldots (ii)$$

Differentiating (i) partially w.r.t. y we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(ax + by + a^2 + b^2\right) = b$$

$$\Rightarrow q = b \quad \ldots (iii)$$

From (i), (ii) & (iii) we get

$$z = px + qy + p^2 + q^2$$

Form partial differential equation from $z=(x^2+a)(y^2+b)$

$$z=(x^2+a)(y^2+b) \quad \ldots(i)$$

Differentiating (i) partially w.r.t. x we get

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left((x^2+a)(y^2+b)\right)=2x(y^2+b)$$

$$\Rightarrow p=2x(y^2+b) \quad \ldots(ii)$$

Differentiating (i) partially w.r.t. y we get

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left((x^2+a)(y^2+b)\right)=2y(x^2+a)$$

$$\Rightarrow q=2y(x^2+a) \quad \ldots(iii)$$

Multiplying (ii) & (iii) we get

$$pq=4xy(x^2+a)(y^2+b)$$

$$\Rightarrow pq=4xyz \qquad (\text{using } (i))$$

Form the Partial Differential Equation by eliminating the arbitrary function

1. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Sol. Given,

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

Differentiating z partially w.r.t. x

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right) \quad \ldots (1)$$

$$\Rightarrow -px^2 = 2f'\left(\frac{1}{x} + \log y\right)$$

$Differentiating\ z\ partially\ w.r.t.\ y$

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$\Rightarrow q - 2y = 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right) \qquad \ldots \qquad (2)$$

$$\Rightarrow qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right)$$

$From\ (1)\ and\ (2)\ we\ get$

$$-px^2 = qy - 2y^2$$

$$\Rightarrow qy + px^2 = 2y^2$$

2. $z = xf_1(x+t) + f_2(x+t)$

Sol. *Given,*

$$z = xf_1(x+t) + f_2(x+t)$$

Differentiating z partially w.r.t. x

$$\frac{\partial z}{\partial x} = f_1(x+t) + xf_1'(x+t) + f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = f_1'(x+t) + f_1'(x+t) + xf_1''(x+t) + f_2''(x+t)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2f_1'(x+t) + xf_1''(x+t) + f_2''(x+t) \qquad \ldots \quad (1)$$

Differentiating z partially w.r.t. t

$$\frac{\partial z}{\partial t} = xf_1'(x+t) + f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial t^2} = xf_1''(x+t) + f_2''(x+t) \qquad \ldots \quad (2)$$

From (1) *and* (2)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 2f_1'(x+t) \qquad \ldots \quad (3)$$

Again

$$\frac{\partial^2 z}{\partial x \partial t} = f_1'(x+t) + xf_1''(x+t) + f_2''(x+t)$$

$$= \frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2}\right) + \frac{\partial^2 z}{\partial t^2} \qquad (\text{Using } (3) \ \& \ (2))$$

3. $xyz = \phi(x+y+z)$

4. $z = f\left(x^2 - y^2\right)$

5. $z = f_1(y - 2x) + f_2(y - 3x)$

6. $f\left(x^2 + y^2, z - xy\right) = 0$