Definations

A differential eq^n involving first order partial derivatives p and q is called a partial differential eq^n of first order.

If both p and q occur in the first degree only and and not multiplied together then it is called linear partial differential eq^n of 1st order.

Lagrange's Linear Equation

The partial differential eq^n of the form

$Pp + Qq = R$ where P, Q, R are f^n of x, y, z is the standard form of a linear partial differential eq^n of first order and is called Lagrange's Linear eq^n.

Lagrange's Auxiliary eq^n or subsidiary eq^n

Lagrange's A. E. :
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Sol^n is given by

$$\phi(u, v) = 0 \quad \text{or} \quad u = f(v)$$

Q. Solve

(1) $\frac{y^2 z}{x} p + xzq = y^2$

Given, $\frac{y^2 z}{x} p + xzq = y^2$

$\Rightarrow y^2 z p + x^2 z q = xy^2$

Here, $P = y^2 z$ $\quad R = xy^2$

$Q = x^2 z$

Now,

Lagrange's A.E. is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2} \quad \text{—— (A)}$$

from (A),

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} \Rightarrow x^2 dx = y^2 dy$$

On integrating both sides we get

$$\int x^2\,dx = \int y^2\,dy$$

$$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1 \quad \text{where } c_1 = \text{int. const}$$

$$\Rightarrow x^3 - y^3 = 3c_1$$

$$\Rightarrow x^3 - y^3 = c \quad \text{where } c = 3c_1 \qquad (1)$$

From (A),

$$\frac{dx}{y^2 z} = \frac{dz}{xy^2}$$

$$\Rightarrow x\,dx = z\,dz$$

On integrating both sides we get

$$\int x\,dx = \int z\,dz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + C_2 \Rightarrow x^2 - z^2 = 2C_2$$

$$\Rightarrow x^2 - z^2 = C' \quad \text{---} \quad (1)$$

$\therefore$ Soln is given by

$$\phi(x^3-y^3,\ x^2-z^2)=0 \quad (\text{Using } ① \ \& \ ⑪)$$

② $pz - qz = z^2+(x^2+y^2)^2$

Here, $P=z$, $Q=-z$, $R=z^2+(x^2+y^2)^2$

Lagrange's A.E. is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2} \quad \text{--- (A)}$$

From (A),

$$\frac{dx}{z} = \frac{dy}{-z} \Rightarrow dx = -dy$$

$$dx + dy = 0$$

on integrating we get

$$\int dx + \int dy = 0$$

$$\Rightarrow x + y = c \quad \text{—— (1)}$$

where c = integrating const.

From (A),

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

$$dx = \frac{z\,dz}{z^2+c^2}$$

let, $t = z^2 + c^2$

$$\frac{dt}{dz} = 2z \Rightarrow \frac{dt}{2} = z\,dz$$

$$dx = \frac{dt/2}{t}$$

$$\Rightarrow 2dx = \frac{dt}{t}$$

On integrating both sides we get,

$$\int 2dx = \int \frac{dt}{t}$$

$$\Rightarrow 2x = \log|t| + c_1$$

$$\Rightarrow 2x = \log|z^2+c^2| + c_1$$

$$\Rightarrow 2x - \log|z^2+c^2| = c_1$$

$$2x - \log\left(z^2 + (x+y)^2\right) = c_1 \quad \text{--- (ii)}$$

$\therefore$ Soln is given by

$$\phi\left(x+y,\ 2x - \log\left(z^2 + x^2 + y^2 + 2xy\right)\right) = 0$$

(Using (i) & (ii))

③ $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

④ $(mz - ny)p + (nx - lz)q = ly - mx$

⑤ $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

Here, $P = z^2 - 2yz - y^2$

$Q = xy + zx$

$R = xy - zx$

Lagrange's A.E. is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx} \quad \text{--- (A)}$$

Taking x, y, z as multipliers we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{\quad}$$

Rough

$$\frac{x\,dx + y\,dy + z\,dz}{(xz^2 - 2xyz - xy^2) + (xy^2 + xyz) + (xyz - z^2x)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0$

On integrating both sides we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$\Rightarrow x^2 + y^2 + z^2 = 2c_1$$

$$\Rightarrow x^2 + y^2 + z^2 = c \quad \text{--- (1)}$$

Again,

$$\frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

$$\Rightarrow \frac{dy}{y+z} = \frac{dz}{y-z}$$

Solve this

Q. $xp + yq = 3z$

Q. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Q. $(y+z)p + (z+x)q = x+y$

Q. $z(xp - yq) = y^2 - x^2$