Non-Linear Eq^n of first order

A P.D.E. which involves 1st order partial derivates p and q with higher degree than one and product of p & q is called non-linear P.D.E. of 1st order.

The complete sol^n consists of two arbitrary constants

(1) Eq^n of the form $f(p,q)=0$

i.e. eq^n involving p & q only and no x, y, z.

consts

Method :

C.S., $z = ax + by + c$ — (1)

$\longrightarrow f(a,b)=0$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

$f(a, b) = 0$

eg

$a + b = 0$

$\Rightarrow b = -a$

$\Rightarrow b = f(a)$

wt, $b = \phi(a)$

① can be written as

$z = ax + \phi(a)y + c \longleftarrow$ C.S

Q. Solve

Rough →

$f(a,b) = 0$

$\Rightarrow \sqrt{a} + \sqrt{b} - 1 = 0$

$\Rightarrow \sqrt{a} + \sqrt{b} = 1$

(a) $\sqrt{p} + \sqrt{q} = 1$

The eqn is of the form $f(p,q) = 0$

$\therefore$ The complete soln is

$z = ax + by + c$ —— (i)

where $\sqrt{a} + \sqrt{b} = 1$

$$\sqrt{a} + \sqrt{b} = 1$$

$$\Rightarrow \sqrt{b} = 1 - \sqrt{a}$$

$$\Rightarrow b = (1 - \sqrt{a})^2$$

sub. b in (1) we get

$$z = ax + (1 - \sqrt{a})^2 y + c$$

where a, c are arb const.

(b) $pq = p+q$

The eq^n is of the form $f(p,q)=0$

$\therefore$ The complete $soln$ is given by

$$z = ax + by + c \quad \text{—— (1)}$$

where $ab = a+b$

$\Rightarrow ab - b = a$

$\Rightarrow b(a-1) = a$

$$b = \frac{a}{a-1}$$

Sub. b in ① we get

$$z = ax + \left(\frac{a}{a-1}\right)y + c$$

where a, c are arb. const.

Q. Solve $\quad x^2p^2 + y^2q^2 = z^2$

$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\Rightarrow \left(\frac{x}{z}p\right)^2 + \left(\frac{y}{z}a\right)^2 = 1$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{——— } (1)$$

Let,

$$\left.\begin{aligned} \frac{dx}{x} &= dX \\ \frac{dy}{y} &= dY \\ \frac{dz}{z} &= dZ \end{aligned}\right\} \qquad \begin{aligned} X &= \log x \\ Y &= \log y \\ Z &= \log z \end{aligned}$$

$$\frac{dZ}{dX} = \frac{x}{z} \cdot \frac{\partial z}{\partial x}, \qquad \frac{dZ}{dY} = \frac{y}{z}\frac{\partial z}{\partial y}$$

From (1)

$$\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dZ}{dY}\right)^2 = 1$$

$$\Rightarrow P^2 + Q^2 = 1$$

where $P = \frac{dZ}{dX}$

$Q = \frac{dZ}{dY}$

which is of the form $f(P, Q) = 0$

The complete soln is given by

$$Z = aX + bY + c$$

where $a^2 + b^2 = 1$

$$\Rightarrow b^2 = 1 - a^2$$

$$\Rightarrow b = \sqrt{1-a^2}$$

$$\therefore Z = aX + \left(\sqrt{1-a^2}\right)Y + c$$

$$\log z = a\log x + \left(\sqrt{1-a^2}\right)\log y + c$$

where a & c are consts.

Eqn of the form $z = px + qy + f(p, q)$

C.S. is $z = ax + by + f(a, b)$

eg: $z = px + qy + \sqrt{1 + p^2 + q^2}$

The given eqn is of the form $z = px + qy + f(p, q)$

$\therefore$ Complete soln is $z = ax + by + \sqrt{1 + a^2 + b^2}$

Q. Solve $4xyz = pq + 2px^2y + 2qxy^2$

Let, $x^2 = X$

$y^2 = Y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y \frac{\partial z}{\partial y}$$

$\therefore$ the given eq^n becomes

$$4xyz = 4xy \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} + 4x^3y \frac{\partial z}{\partial X} + 4xy^3 \frac{\partial z}{\partial Y}$$

$$\Rightarrow z = x^2 \frac{\partial z}{\partial X} + y^2 \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

$$\Rightarrow z = X \frac{\partial z}{\partial X} + Y \frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

$$\Rightarrow z = PX + QY + PQ$$

where $P = \dfrac{\partial z}{\partial X}$

$Q = \dfrac{\partial z}{\partial Y}$

$\therefore$ Complete soln is $z = aX + bY + ab$

$= ax^2 + by^2 + ab$

Q1. $z = px + qy - 2\sqrt{pq} \longrightarrow z = ax + by - 2\sqrt{ab}$

Q2. $z = px + qy + \sin(p+q) \longrightarrow z = ax + by + \sin(a+b)$

Q3. $(pq - p - q)(z - px - qy) = pq$

Eqn of the form $f(z, p, q) = 0$ (it will not contain x & y)

Algorithm :

ⓐ Assume $u = x + ay$

$$\Rightarrow p = \frac{\partial z}{\partial u}$$

$$\& \quad q = a\frac{\partial z}{\partial u}$$

ⓑ sub. p & q in given eqn

ⓒ given eq $= f(z, u)$

ⓓ Replace u with $x + ay$

soln will be

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial y}\right) = 0$$

Not read

$z = f(x+ay)$

$$p = \frac{\partial z}{\partial x}$$

$$= f'(x+ay)$$

$$= f'(u)$$

$$= \frac{\partial z}{\partial u}$$

Q. Solve $z^2(p^2+q^2+1)=a^2$

The given eq^n is of the form

$$f(z,p,q)=0$$

let, $u = x + a_1 y$

$$p = \frac{\partial z}{\partial u}$$

$$q = a_1 \frac{\partial z}{\partial u}$$

sub. p & q in the given eqⁿ we get

$$z^2\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(a_1\frac{\partial z}{\partial u}\right)^2+1\right]=a^2$$

$$\Rightarrow z^2\left[\left(\frac{\partial z}{\partial u}\right)^2+a_1^2\left(\frac{\partial z}{\partial u}\right)^2+1\right]=a^2$$

$$\Rightarrow z^2\left[(1+a_1^2)\left(\frac{\partial z}{\partial u}\right)^2\right]+z^2=a^2$$

$$\Rightarrow z^2(1+a_1^2)\left(\frac{\partial z}{\partial u}\right)^2=a^2-z^2$$

$$\Rightarrow z\sqrt{1+a_1^2}\ \frac{\partial z}{\partial u} = \pm\sqrt{a^2-z^2}$$

$$\Rightarrow z\sqrt{1+a_1^2}\ dz = \pm\sqrt{a^2-z^2}\ du$$

$$\Rightarrow \pm\frac{z\sqrt{1+a_1^2}}{\sqrt{a^2-z^2}}\,dz = du$$

On integrating we get

$$\int\left\{\pm\frac{z\sqrt{1+a_1^2}}{\sqrt{a^2-z^2}}\right\}dz = \int du$$

$$\Rightarrow \pm\sqrt{1+a_1^2}\int \frac{z}{\sqrt{a^2-z^2}}\,dz = \int du$$

$t = a^2 - z^2$

$$\Rightarrow \pm\sqrt{1+a_1^2}\,\sqrt{a^2-z^2} = u + c$$

where c = integrating constant

$$\Rightarrow \pm\sqrt{1+a_1^2}\,\sqrt{a^2-z^2} = (x+a_1 y) + c$$

$$\Rightarrow \left(\sqrt{1+a_1^2}\,\sqrt{a^2-z^2}\right)^2 = \left((x+a_1 y)+c\right)^2$$

$$\Rightarrow (1+a_1^2)(a^2-z^2) = (x+a_1 y+c)^2$$

Equations of the form $f_1(x, p) = f_2(y, q)$

Eqns in which z is absent and terms involving x and p can be seperated from those involving y and q.

Method: $f_1(x, p) = f_2(y, q) = a$ where a = const.

let, $p = F_1(x)$

$q = F_2(y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy$$

$$dz = F_1(x)dx + F_2(y)dy$$

$$\Rightarrow z = \int F_1(x)dx + \int F_2(y)dy$$

which the reqd C.S.

Q. solve $p^2 - q^2 = x - y$

Given, $p^2 - q^2 = x - y$

$\Rightarrow p^2 - x = q^2 - y$ which is of the form

$$f_1(x, p) = f_2(y, q)$$

Let, $p^2 - x = q^2 - y = a$

$\Rightarrow p^2 - x = a$ and $q^2 - y = a$

$\Rightarrow p = \sqrt{x+a}$ and $q = \sqrt{y+a}$

Sub. p & q in $dz = p\,dx + q\,dy$

$\Rightarrow dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$

On integrating we get,

$z = \int \sqrt{x+a}\,dx + \int \sqrt{y+a}\,dy$

$\therefore\ z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c$ where c = const.

Q. Solve $yp = 2yx + \log q$

Given, $yp = 2yx + \log q$

$\Rightarrow p = 2x + \frac{1}{y}\log q$

$\Rightarrow p - 2x = \frac{1}{y}\log q$

which is of the form $f_1(x, p) = f_2(y, q)$.

let, $p - 2x = \frac{1}{y} \log q = a$

$\Rightarrow p - 2x = a$ and $\frac{1}{y} \log q = a$

$\Rightarrow p = 2x + a$ and $\log q = ya$

$\Rightarrow q = e^{ya}$

sub. p & q in $dz = p\,dx + q\,dy$

$\Rightarrow dz = (2x+a)\,dx + e^{ya}\,dy$

On integrating we get,

$z = \int (2x+a)\,dx + \int e^{ya}\,dy = x^2 + ax + \frac{1}{a} e^{ya} + c$

where c = const.

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

Q. Solve $\quad z^2(p^2+q^2) = x^2+y^2$

$$z^2p^2 + z^2q^2 = x^2+y^2$$

$$\Rightarrow (zp)^2 + (zq)^2 = x^2+y^2$$

$$\Rightarrow \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2 \longrightarrow (1)$$

let, $\quad Z = \frac{1}{2}z^2$

$$\Rightarrow \frac{dZ}{dz} = z \quad \Rightarrow dZ = z\,dz$$

Now,

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = z \frac{\partial z}{\partial x}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = z \frac{\partial z}{\partial y}$$

Sub. the above in eqn (1)

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$$

$$\Rightarrow P^2 + Q^2 = x^2 + y^2$$

$$\Rightarrow P^2 - x^2 = y^2 - Q^2$$ which is of the form $f_1(x, P) = f_2(y, Q)$

let, $p^2 - x^2 = y^2 - Q^2 = a$

$\Rightarrow p^2 - x^2 = a$ and $y^2 - Q^2 = a$

$\Rightarrow P = \sqrt{x^2 + a}$ and $Q = \sqrt{y^2 - a^2}$

sub. p & Q in $dZ = P dx + Q dy$

$\Rightarrow dZ = \sqrt{x^2 + a}\, dx + \sqrt{y^2 - a}\, dy$

On integrating we get

$Z = \int \sqrt{x^2 + a}\, dx + \int \sqrt{y^2 - a}\, dy$

$$Z = \frac{1}{2} x\sqrt{x^2+a} + \frac{a}{2}\log\left(x+\sqrt{x^2+a}\right) +$$

$$\frac{1}{2} y\sqrt{y^2-a} - \frac{a}{2}\log\left(y+\sqrt{y^2-a}\right) + c$$

where $c = \text{const}$.

$$\Rightarrow \frac{1}{2}Z^2 = \frac{1}{2} x\sqrt{x^2+a} + \frac{a}{2}\log\left(x+\sqrt{x^2+a}\right) +$$

$$\frac{1}{2} y\sqrt{y^2-a} - \frac{a}{2}\log\left(y+\sqrt{y^2-a}\right) + c$$

$$\Rightarrow Z^2 = x\sqrt{x^2+a} + a\log\left(x+\sqrt{x^2+a}\right) + y\sqrt{y^2-a} - a\log\left(y+\sqrt{y^2-a}\right) + c$$

Q1. $q = xyp^2$

Q2. $yp + xq + pq = 0$

Q3. $z(p^2 - q^2) = x - y$

Q4. $p + q = \sin x + \sin y$