# Normal Distribution

## Probability Density $F^n$ :

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-1}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty \le x \le \infty$$

$$\sigma > 0$$

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \mu$$

$$Var(X) = E[X-\mu]^2 = E[X^2] - (E[X])^2$$

$$= \sigma^2$$

$$S.D. = \sigma$$

eg: pdf of ~~of~~ a normal distribution is

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}$$

$\therefore \mu = 2$

$\sigma = 3$

Q. Prove that median of normal distribution is $\mu$. (Submission : Oct 20, 2020) (5)

Q. Prove that mode of normal distribution is $\mu$. (Submission : Oct 20, 2020) (5)

* For normal distribution, mean = median = mode

## Properties

① $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty \leq x \leq \infty$


② The curve is bell-shaped and symmetrical about $\mu$.

③ Mean, median & mode of normal dist. coincide.

④ The max$^m$ prob. occurs at the point $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$

(5) Mean deviation about mean $= \frac{4\sigma}{5}$

(6) No portion of curve lies in negative x-axis (As prob. cannot be -ve)

(7) The point of inflextion are at $x = \mu \pm \sigma$

(1) Area of the normal curve betn $(\mu - \sigma)$ and $(\mu + \sigma)$ is 0.6826

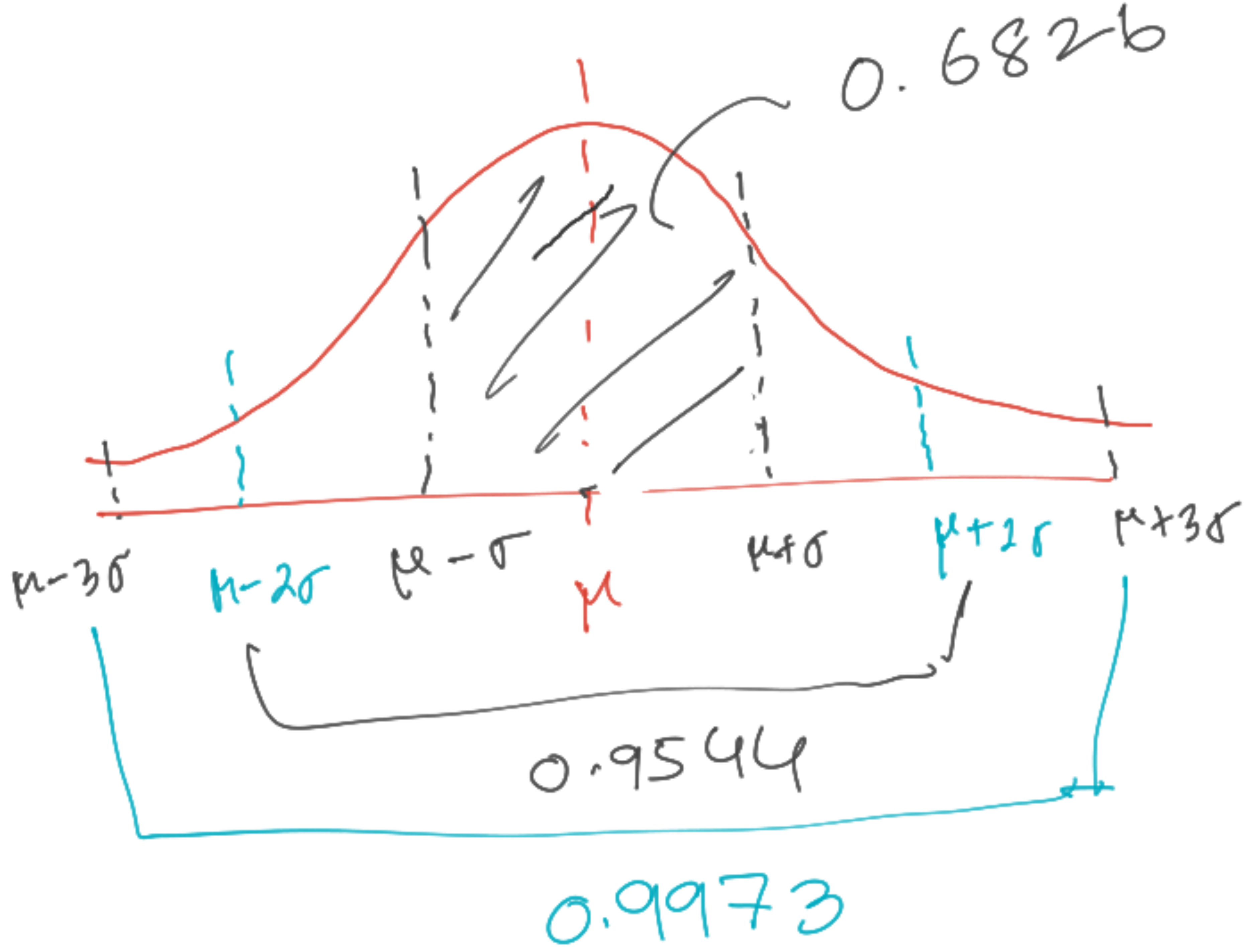
$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$

(ii) Area of the normal curve in betⁿ $(\mu-2\sigma)$ and $(\mu+2\sigma)$ is 0.9544

$P(\mu-2\sigma < X < \mu+2\sigma) = 0.9544$

(iii) Area of the normal curve in betⁿ $(\mu-3\sigma)$ and $(\mu+3\sigma)$ is 0.9973

$P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$



0.6826
μ-3σ
μ-2σ
μ-σ
μ
μ+σ
μ+2σ
μ+3σ
0.9544
0.9973

## Conditions

① The no. of trials is large, $n \to \infty$

② Neither p nor q is too small, $p \to q$

Poisson

$n \to \infty$

$p \to 0$ very small

$q \to 1$ very large

Normal

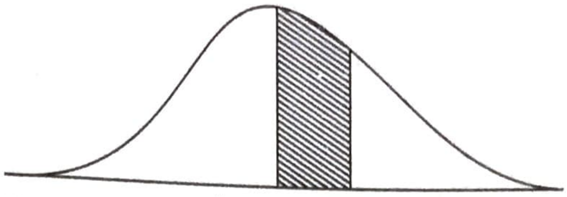
$n \to \infty$

$p \to q$

i.e. p & q both are

<u>not</u> very small

**Table 3. Areas under the Standard Normal Curve from 0 to $\frac{x-\mu}{\sigma}$**



| *t or z* | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0 0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0745 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0348 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1808 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2258 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2218 | 0.2549 |
| 0.7 | 0.2580 | 0.2612 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2996 | 0.3023 | 0.3051 | 0.3078 | 03106 | 0.3133 |
| 0.9 | 03159 | 0.3186 | 0.3212 | 03238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3980 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.7418 | 0.4429 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4484 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4699 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4756 | 0.4761 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4864 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.1916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 40.4978 | 0.4879 | 0.4979 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4982 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986. |
| 3.0 | 0.4987 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |
| 3.1 | 0.4990 | 0.4991 | 0.4991 | 0.4991 | 0.4992 | 0.4992 | 0.4992 | 0.4992 | 0.4993 | 0.4993 |
| 3.2 | 0.4993 | 0.4993 | 0.4994 | 0.4994 | 0.4994 | 0.4994 | 0.4995 | 0.4995 | 0.4995 | 0.4995 |
| 3.3 | 0.4995 | 0.4995 | 0.4995 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4996 | 0.4997 |
| 3.4 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4997 | 0.4998 |
| 3.5 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 | 0.4998 |
| 3.6 | 0.4998 | 0.4998 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.7 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.8 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 | 0.4999 |
| 3.9 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |

Variate ⟷ X

Normal Variate ⟷ Z

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{x - \mu}{\sigma}$$

Q. If $\mu = 50$ and $\sigma = 10$ find

(i) $P(50 \le X \le 80)$

(ii) $P(60 \le X \le 70)$

(iii) $P(30 \le X \le 40)$

(iv) $P(40 \le X \le 60)$

$3.0 \longrightarrow 3.09$

Here, $\mu = 50$

$\sigma = 10$

we know that, $Z = \dfrac{X - \mu}{\sigma} \longleftarrow$

(1) $P(50 \leq X \leq 80)$

$= P\left(\dfrac{50-50}{10} \leq Z \leq \dfrac{80-50}{10}\right)$

$= P(0 \leq Z \leq 3) = 0.4987$

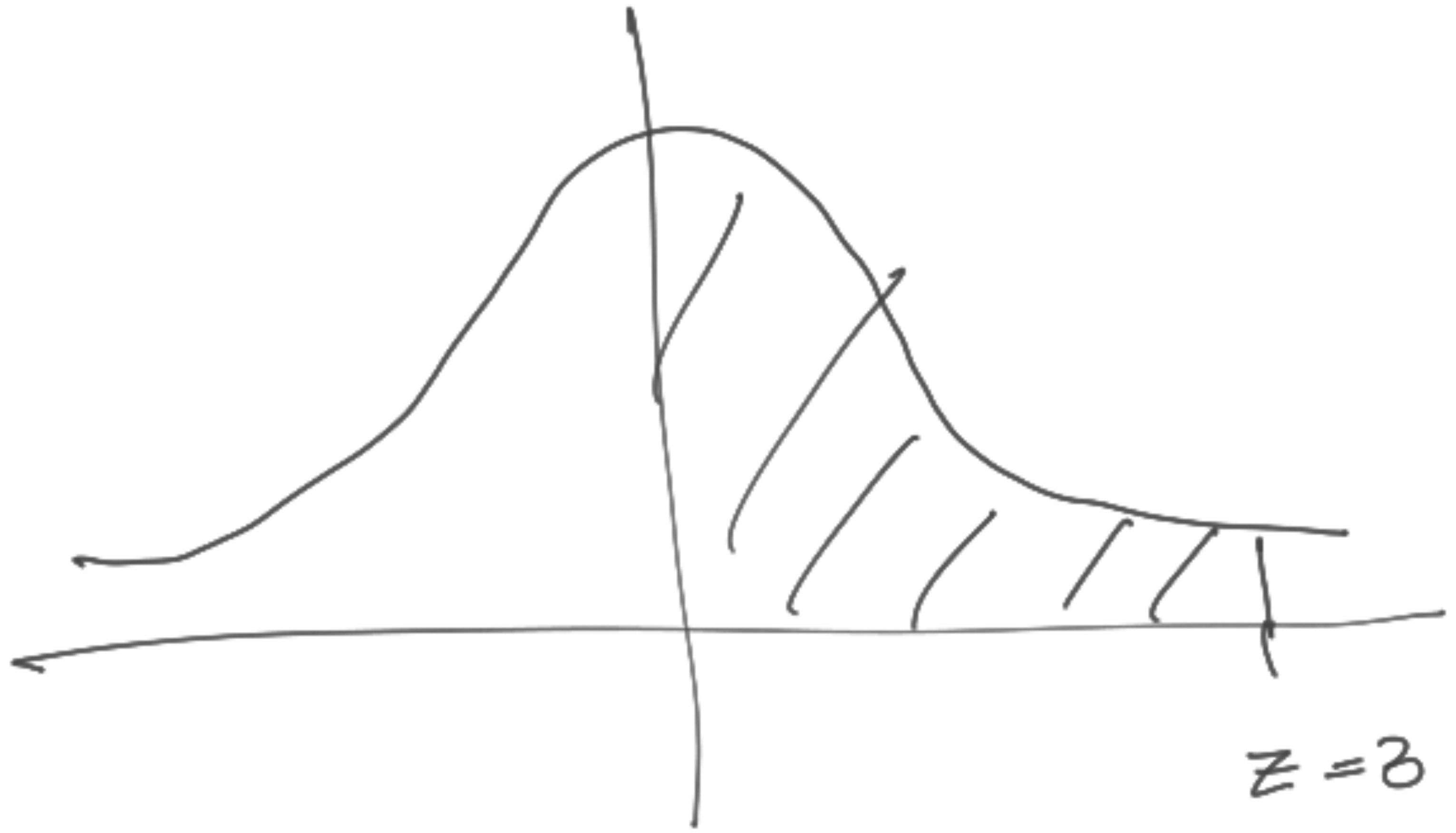
Rough $\rightarrow$ ($\longrightarrow$ Area from $Z=0$ to $Z=3$)

$50 \leq X \leq 80$

$\dfrac{50-50}{10} \leq \dfrac{X-\mu}{\sigma} \leq \dfrac{80-50}{10}$

$\Rightarrow 0 \leq Z \leq 3$

$P(0 \le Z \le 3)$



$z = 3$

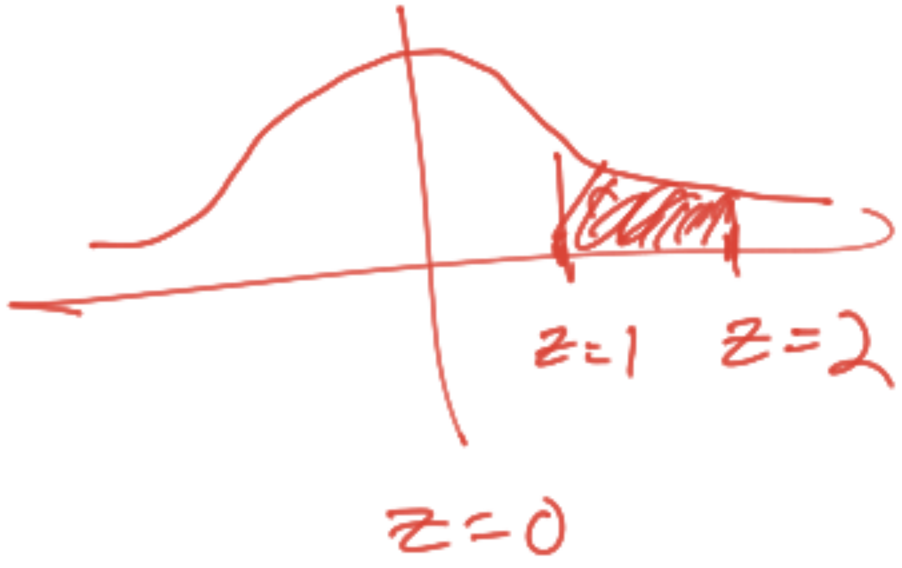
$Z = 0$ $\quad$ $x = 80$

$x = \mu = 50$

(ii) $P(60 \le X \le 70)$

$$= P\left(\frac{60-50}{10} \le Z \le \frac{70-50}{10}\right)$$

$$= P(1 \le Z \le 2)$$




= (Area from z = 0 to z = 2)

 − (Area from z = 0 to z = 1)

$$= 0.4772 - 0.3413 = 0.1359$$

(iii) $P(30 \le X \le 40)$

$$= P\left(\frac{30-50}{10} \le Z \le \frac{40-50}{10}\right)$$

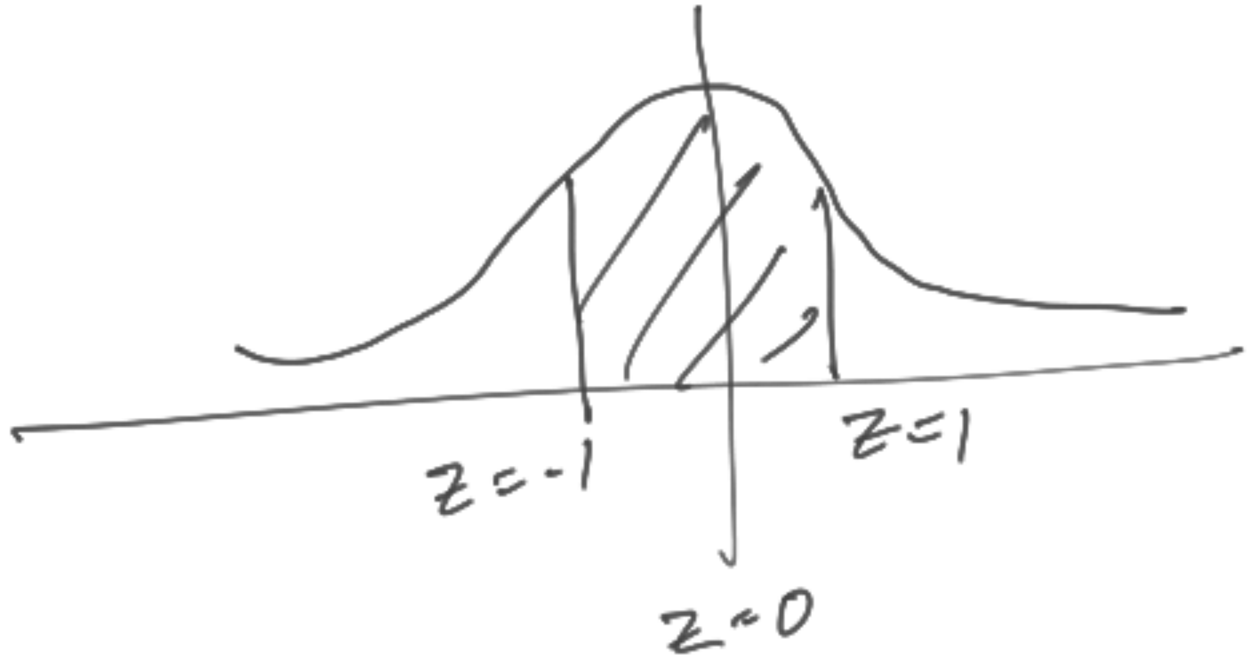

$$= P(-2 \le Z \le -1)$$

$$= P(1 \le Z \le 2)$$ ($\because$ The curve is symmetrical)

$$= 0.1359$$

(iv) $P(40 \le X \le 60)$

$= P\left(\frac{40-50}{10} \le Z \le \frac{60-50}{10}\right)$




$= P(-1 \le Z \le 1)$

= Area from $z=-1$ to $z=1$

$= 2 \times$ (Area from $z=0$ to $z=1$)

$= 2 \times 0.3413 = 0.6826$

Q. A sample of 100 dry battery cells tested to find the length of life produced the following results

$$\mu = 12 \text{ hrs}, \quad \sigma = 3 \text{ hrs}$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

(a) more than 15 hrs

(b) less than 6 hrs

(c) between 10 and 14 hrs.

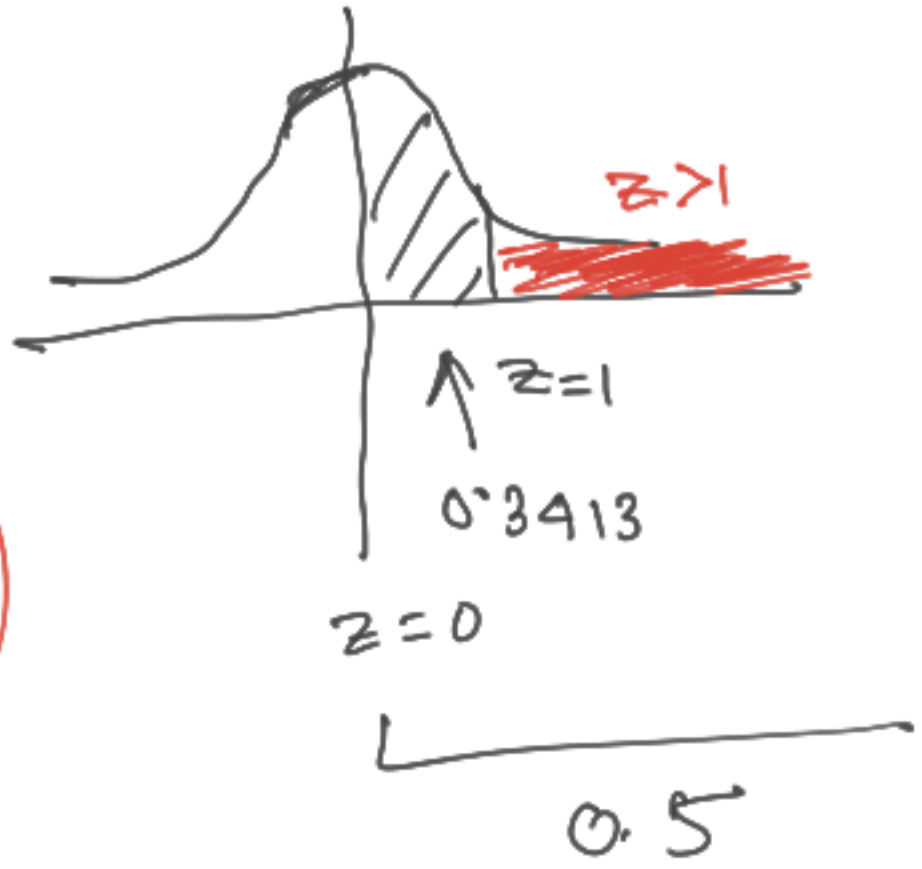
$$Z = \frac{X - \mu}{\sigma}$$

normal variate

$$= \frac{X - 12}{3}$$

x denotes the length of life of dry battery cells

$$Z = \frac{X-\mu}{\sigma} = \frac{X-12}{3}$$



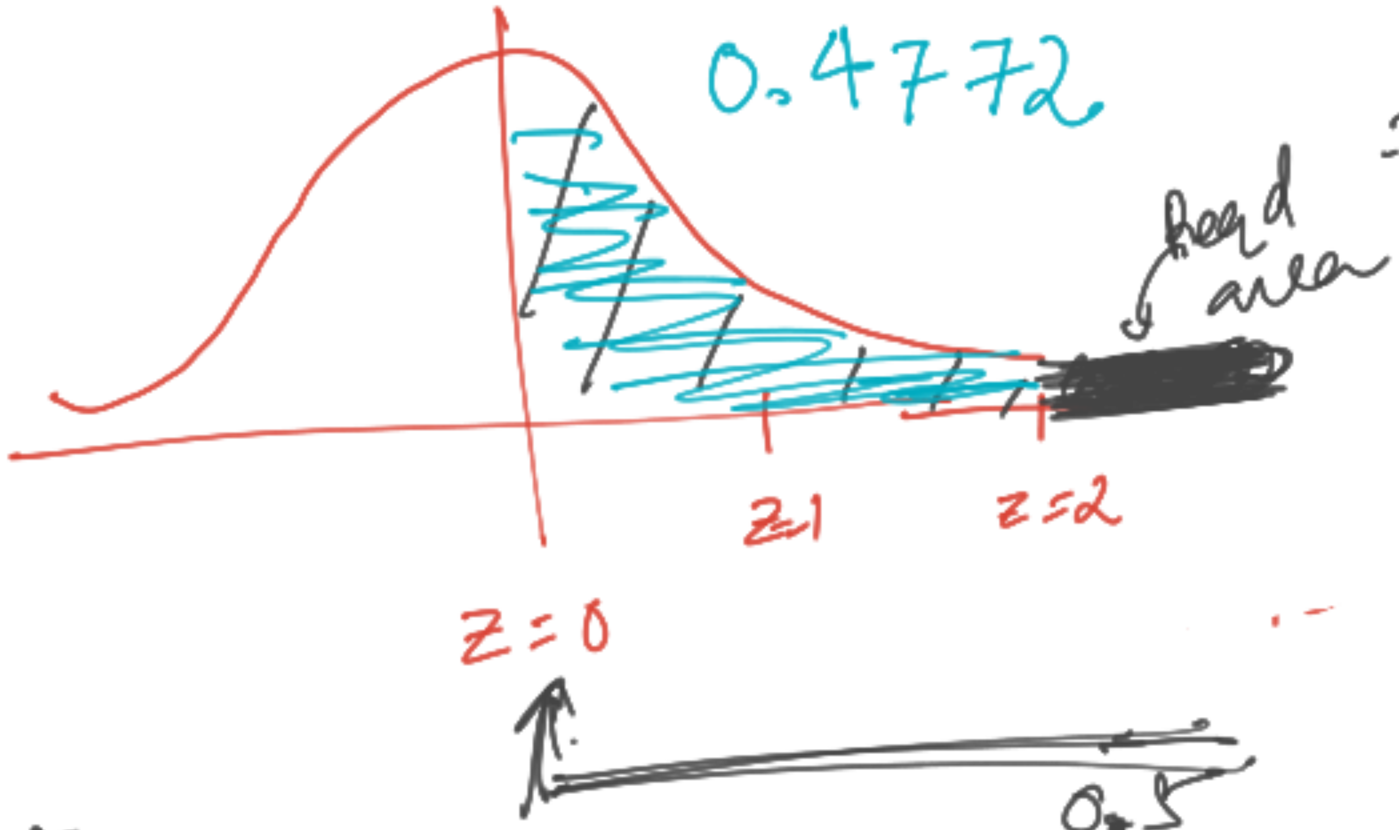

(a) $$P(X>15) = P\left(\frac{X-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right)$$

$$= P\left(Z > \frac{15-12}{3}\right)$$

$$= P(Z>1) = 0.5 - 0.3413 = 0.1587$$

∴ Percentage of battery cells having life more than 15 hrs = $0.1587 \times 100$ $= 15.87\%$

(b) $P(X < 6) = P\left(\frac{X-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right)$




Total area = 1

$= P\left(Z < \frac{6-12}{3}\right)$

$= P(Z < -2)$

$= P(Z > 2)$ ($\because$ The area is symmetrical)

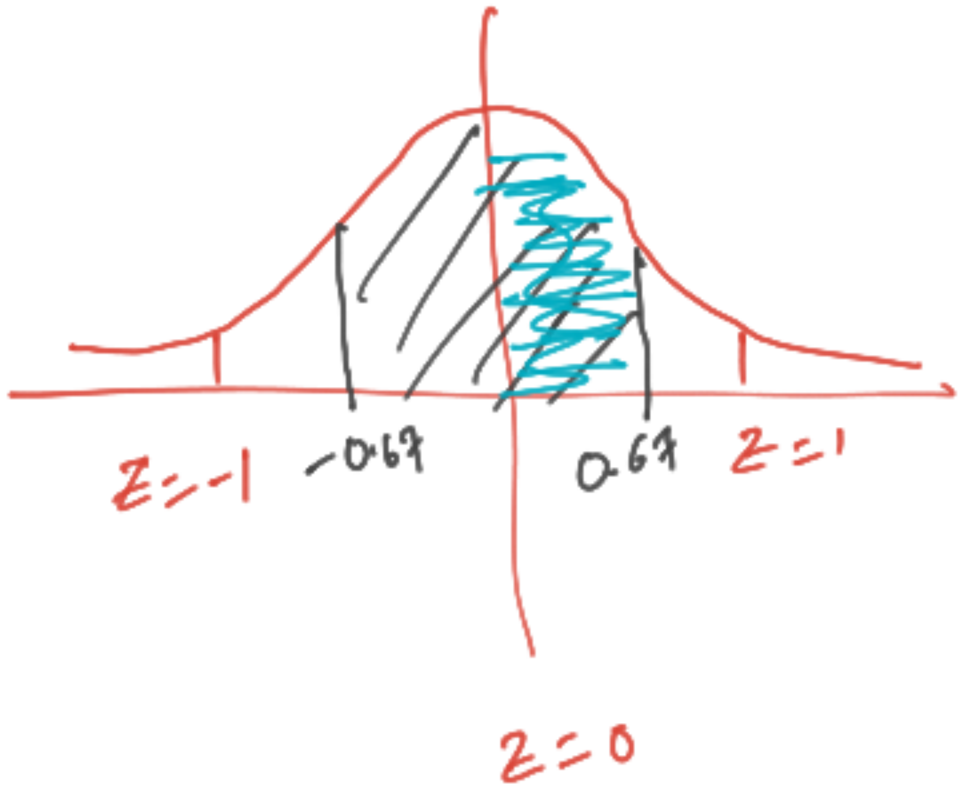
$= 0.5 - 0.4772$

$= 0.0229$

$\therefore$ Percentage of battery cells having life less than 6hrs $= 0.0229 \times 100$

$= 2.29\%$

(c) $P(10 < X < 14) = P\left(\frac{10-12}{3} < Z < \frac{14-12}{3}\right)$




$= P(-0.67 < Z < 0.67)$

$= 2 \times P(0 < Z < 0.67)$

$= 2 \times 0.2486$

$= 0.4972$

$\therefore$ Percentage of battery cells having life span between 10 hrs and 14 hrs = 49.72%

Q. The average height of soldiers of a country is given as 68.22 inches with variance 10.8 sq. inch. How many soldiers out of 1000 would you expect to be over 72 inches tall? Given that the area under the normal curve between $z = 0$ to $z = 0.35$ is 0.1368 and betn $z = 0$ and $z = 1.15$ is 0.3746.

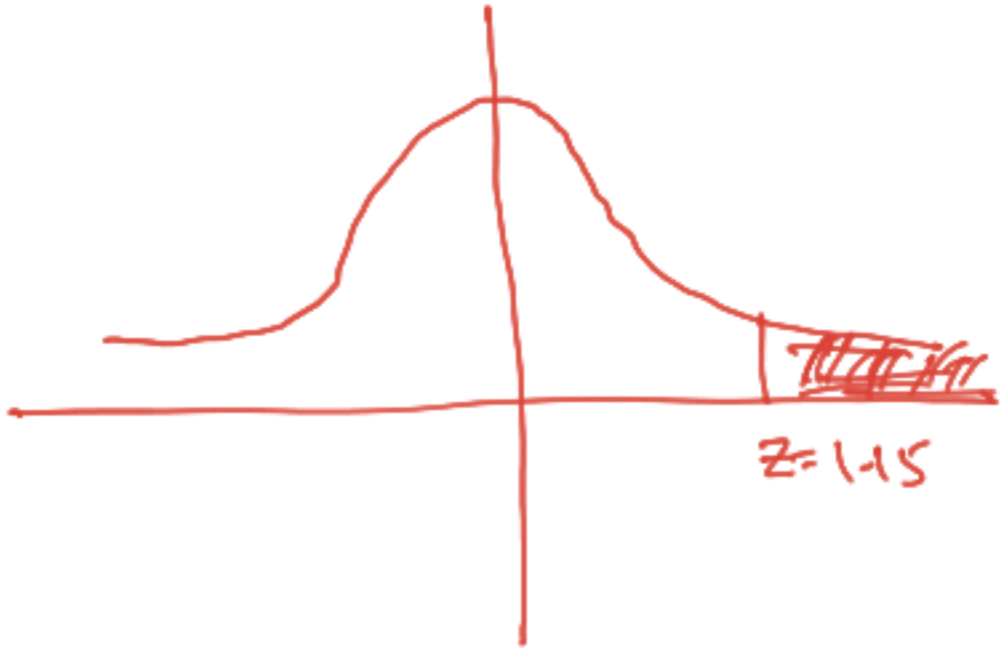
Here, $\mu = 68.22$

$\sigma = \sqrt{10.8}$

X denotes the height of the soldier in inches

Now,

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 68.22}{\sqrt{10.8}}$$




$$P(X \geq 72) = P\left(Z \geq \frac{72 - 68.22}{\sqrt{10.8}}\right)$$

$$= P(Z \geq 1.15)$$

$$= 0.5 - 0.3749 = 0.1251$$

∴ No. of soldiers expected to have height more than 72 inches

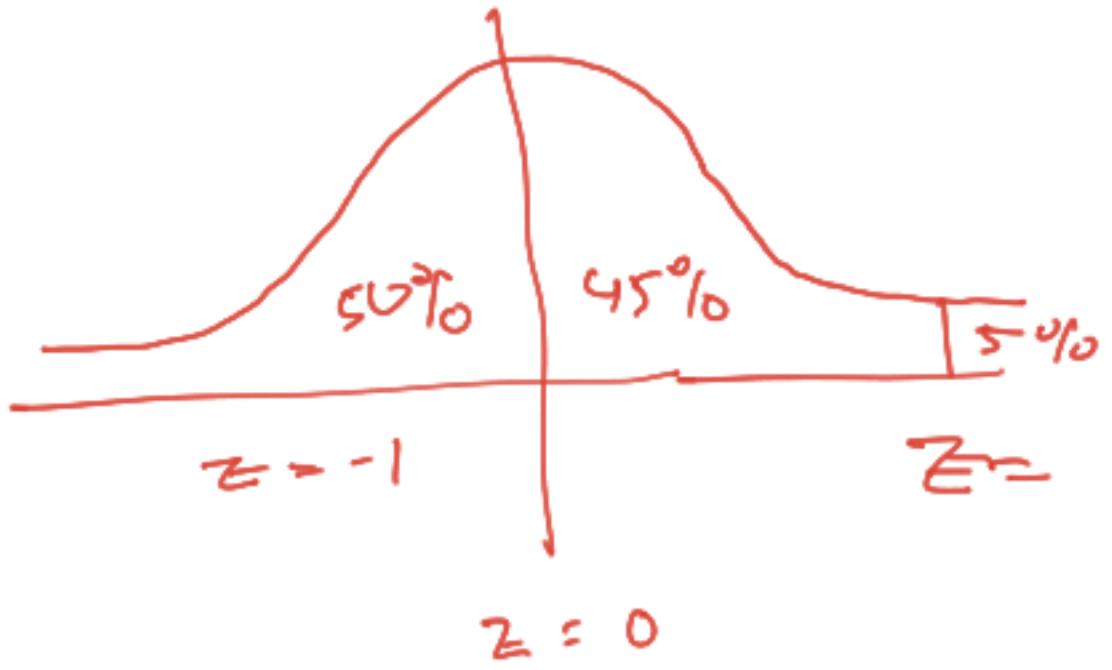
approx.

$= 100 \times 0.1251 = 12.5$

Q. The distribution of a random variable is given by

$$f(x) = Ce^{-\frac{1}{50}(9x^2 - 30x)}, \quad -\infty < x < \infty$$

Find constant $C$, the mean and the variance of the random variable. Find also the upper 5% value of the random variable.



$$f(x) = Ce^{-\frac{1}{50}\left((3x)^2 - 2\cdot 3x\cdot 5) + 5^2 - 5^2\right)}$$

$$= ce^{-\frac{9}{50}\left[\left(x - \frac{5}{3}\right)^2 - \frac{25}{9}\right]}$$

$$= ce^{-\frac{9}{2}\left[\left(\frac{x-5/3}{5}\right)^2 - \frac{1}{25}\cdot\frac{25}{9}\right]}$$

$$= c\cdot e^{-9/2\left[\left(\frac{x-5/3}{5}\right)^2\right] + \frac{9}{2}\cdot\frac{1}{9}}$$

$$= ce^{-\frac{9}{2}\left[\left(\frac{x-5/3}{5}\right)^2\right]} \cdot e^{\frac{1}{2}}$$

$$= ce^{1/2}\, e^{-\frac{1}{2}\left[\left(\frac{x-5/3}{5/3}\right)^2\right]}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f(x) = Ce^{-\frac{1}{50}(9x^2 - 30x)}$$

Here, $\mu = 5/3$

$\sigma = 5/3$

Variance $= \sigma^2 = \frac{25}{9}$

Mean $= \mu = 5/3$

Again,

$$ce^{1/2} = \frac{1}{\sigma\sqrt{2\pi}}$$

$$\Rightarrow ce^{1/2} = \frac{1}{5/3\sqrt{2\pi}} \Rightarrow c = \frac{3}{5\sqrt{2\pi e}}$$

$$= 0.145$$

S.D ← Standard Deviation

Assuming that the diameters of 1000 brass plugs taken consecutively from a machine form a normal dist. with the mean 0.7515cm and S.D. 0.002cm. Find the number of plugs likely to be rejected if the approve diameter is 0.752 ± 0.004 cm.

Given,

$\mu = 0.7515\,cm$

$S.D = 0.002\ cm$

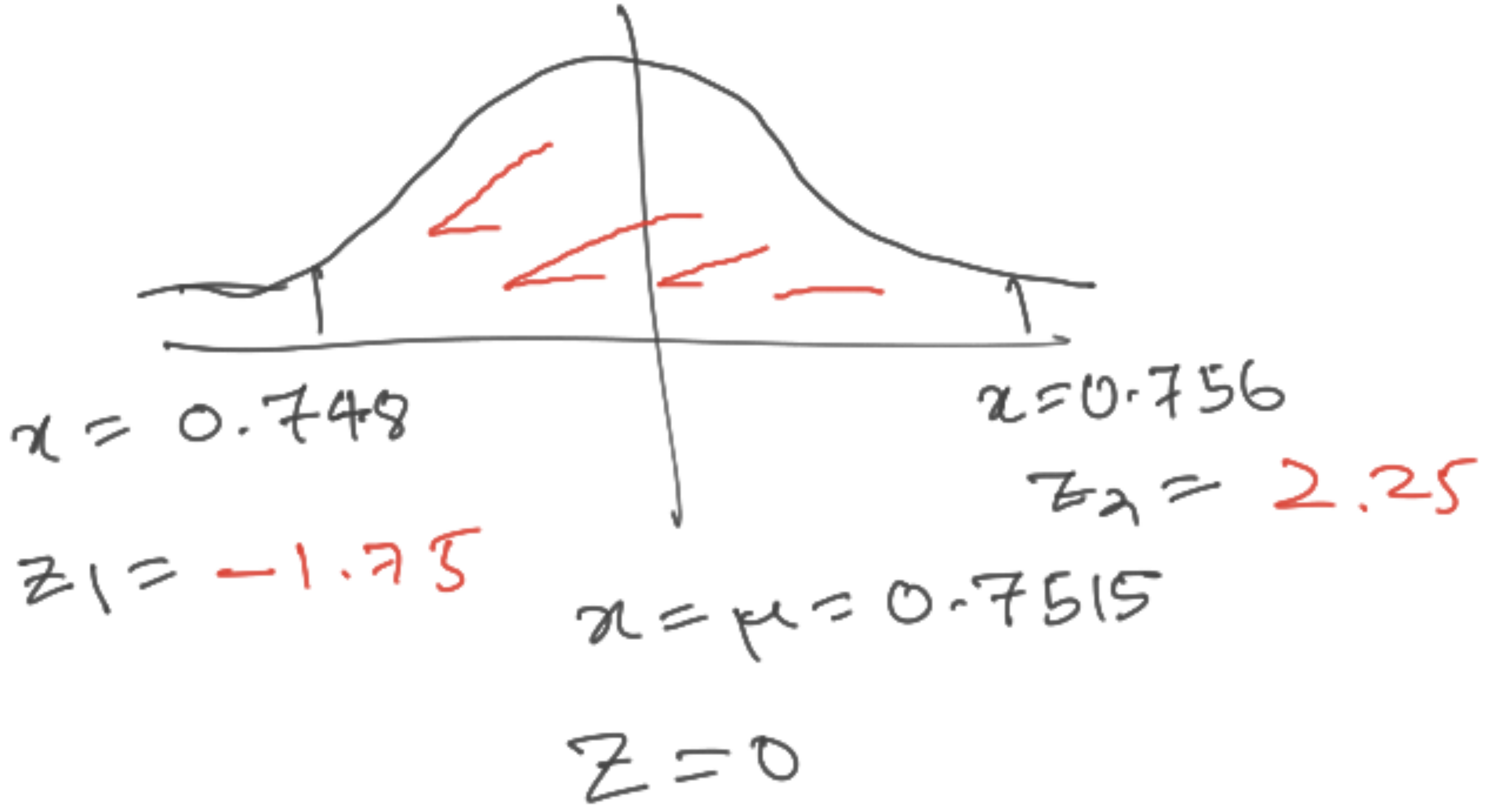
Limits of diameter of non-defective plugs are
0.752 + 0.004 = 0.756 & 0.752 − 0.004 = 0.748

At $x = 0.748$,

$$z_1 = \frac{x - \mu}{\sigma}$$

$$= \frac{0.748 - 0.7515}{0.002}$$

$$= -1.75$$




At $x = 0.756$

$$z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$$

$P(-1.75 \leq Z \leq 2.25) = $ (Area from $Z=0$ to $Z=1.75$)

+ (Area from $Z=0$ to $Z=2.25$)

$= 0.4599 + 0.4878$

$= 0.9477$

No. of Non-defective plugs $= 1000 \times 0.9477$

$= 947.7$

$= 948$

No. of defective plugs $= 1000 - 948 = 52$

Q. Suppose 10% of probability for a normal dist. $N(\mu, \sigma^2)$ is below 35 and 5 percent above 90 what are the value of $\mu$ and $\sigma$

$$P(x < 35) = \frac{10}{100} = 0.1$$

$$P(x > 90) = \frac{5}{100} = 0.05$$

Now,

$$z = \frac{x - \mu}{\sigma}$$

When $x = 35$

$$P(x < 35) = 0.1$$

$$\Rightarrow P\left(Z < \frac{35 - \mu}{\sigma}\right) = 0.1$$

When $x = 90$,

$$P(x > 90) = 0.05$$

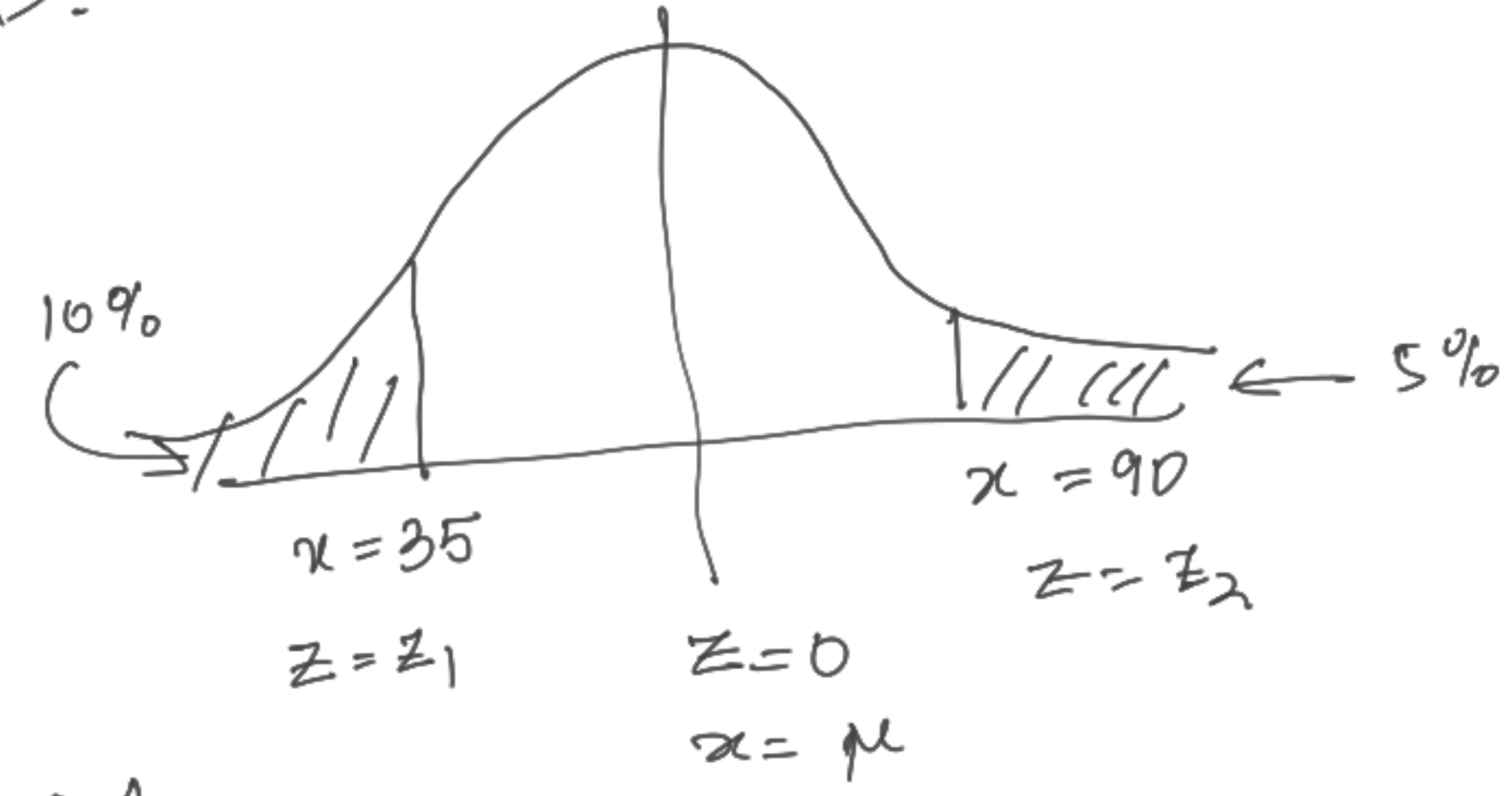
$$\Rightarrow P\left(Z > \frac{90-\mu}{\sigma}\right) = 0.05$$

Q. Suppose 10% of probability for a normal distribution $N(\mu, \sigma^2)$ is below 35 and 5% above 90, what are the value of $\mu$ and $\sigma$.

let $\mu$ = mean & $\sigma$ = S.D.

$$P(x < 35) = \frac{10}{100} = 0.1$$

$$P(x > 90) = \frac{5}{100} = 0.05$$




when $x = 35$, let $z = z_1$

$$P(z_1 < z < 0) = 0.5 - 0.1 = 0.4$$

$\therefore Z_1 = -1.29$

when $x = 90$, let $Z = Z_2$

$$P(0 < Z < Z_2) = 0.5 - 0.05 = 0.45$$

$$\Rightarrow Z_2 = 1.65$$

Now, $Z_1 = -1.29 \Rightarrow \frac{35 - \mu}{\sigma} = -1.29$

$$\Rightarrow 35 - \mu = -1.29\sigma \quad \text{--- (1)}$$

$z_2 = 1.65 \Rightarrow \dfrac{90-\mu}{\sigma} = 1.65$

$\Rightarrow 90 - \mu = 1.65\sigma$ —— (ii)

Solving (i) & (ii)

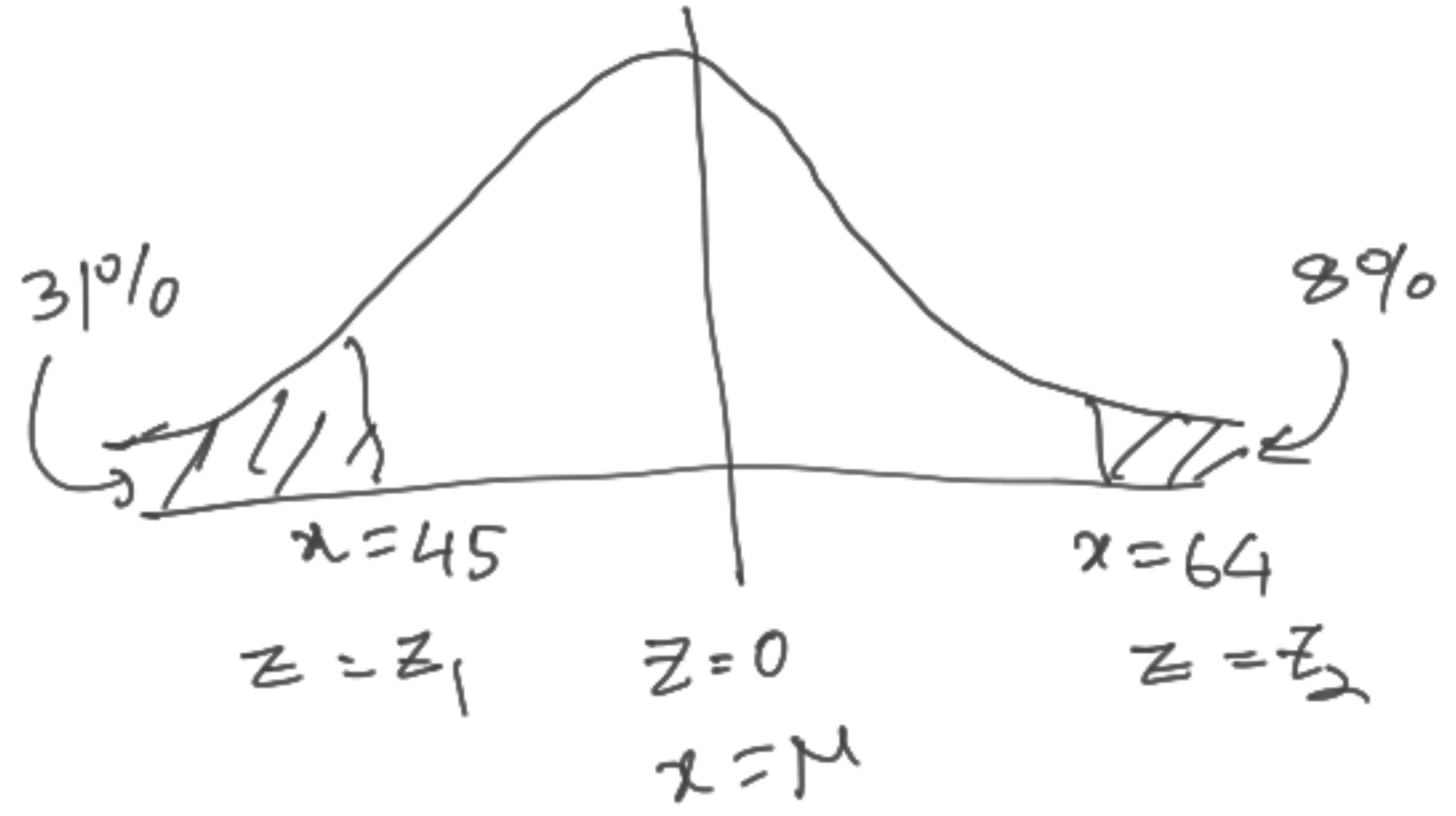
$\mu = 59.133$

$\sigma = 18.707$

Q. In a normal distribution, 31% of the items are under 45 and 8% over 64. Find the mean and S.D.

let, mean be $\mu$

and S.D. be $\sigma$




Ans

$\mu = 50$ (approx)

$\sigma = 10$ (approx)