Bivariate Distribution

Joint Probability

$$P_{XY}(x,y) \quad \text{or} \quad f_{XY}(x,y)$$

$\downarrow$

Joint prob. of two random variables

X & Y

Joint Probability Mass Function

X and Y be two R.V. on a sample space S with respect to image sets $X(S) = \{x_1, x_2, \ldots, x_n\}$ and $Y(S) = \{y_1, y_2, \ldots, y_m\}$

Function p on $X(S) \times Y(S)$ defined as

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$

is called joint probability mass f^n of X & Y where

$$X(S) \times Y(S) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$$

$x \backslash y$	y_1	y_2	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	$\cdots$	p_{1m}	p_1
x_2	p_{21}	p_{22}	$\cdots$	p_{2m}	p_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\cdots$	p_{nm}	p_n
Total	p'_1	p'_2	$\cdots$	p'_m	

p_1 or p'_1

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Marginal and Conditional probability Function

Marginal Probability f^n of X :

$$f_X(x) = p_X(x_i) = P(X = x_i) = p_{i1} + p_{i2} + \dots + p_{im}$$

$$= \sum_{j=1}^{m} p_{ij}$$

$$= p_{i\circ}$$

Marginal Probability f^n of Y :

$$f_Y(y) = p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = p_{\circ j}$$

Conditional Probability F^n of X when $Y = y_j$ is given

$$f_{X/Y}(x/y) = P(X = x_i / Y = y_j) = \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

$$= \frac{p(x_i, y_j)}{p(y_j)}$$

$$= \frac{p_{ij}}{p_j}$$

$$P(X = x_1 / Y = y_3)$$

$$= \frac{p_{13}}{p_3}$$

Conditional Prob. fn of Y when $X = x_i$ is given

$$f_{Y/X}(y/x) = P(Y = y_j / X = x_i) = \frac{p(x_i, y_j)}{p(x_i)}$$

$$= \frac{p_{ij}}{p_i}$$

Independent :— $P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$

Joint Probability Distribution Function

(X,Y) be two dimensional R.V. then their Joint Probability Dist. F^n is denoted by $F_{X,Y}(x,y)$ and it is defined as

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) \quad \forall \; x,y \in \mathbb{R}$$

where, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1 \quad \leftarrow$ cont. dist.

$\sum_x \sum_y f_{X,Y}(x,y) = 1 \quad \leftarrow$ discrete dist.

Properties

(1) If $x_1 < x_2$ and $y_1 < y_2$

$$P[x_1 < X \leq x_2 ,\ y_1 < Y \leq y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geqslant 0$$

(2) (a) $F(-\infty, y) = \lim\limits_{x \to \infty} F(x, y) = 0 \quad \forall \ y \in R$

(b) $F(x, -\infty) = \lim\limits_{y \to \infty} F(x, y) = 0 \quad \forall \ x \in R$

(c) $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = F(\infty, \infty) = 1$

③ $F(x,y)$ is right continuous in each argument

$$\lim_{h \to 0^+} F(x+h, y) = \lim_{h \to 0^+} F(x, y+h) = F(x,y)$$

④ If the density f^n $f(x,y)$ is continuous at (x,y) then

$$\frac{\partial^2 F}{\partial x \partial y} = f(x,y)$$

Expectation

(X,Y) be two dimensional R.V. with Joint density f^n $f_{X,Y}(x,y)$. The expectation of $g(X,Y)$ is denoted by $E(g(X,Y))$ and defined by

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, f_{X,Y}(x,y)$$

Note

① $E[X] = \sum x f_X(x,y)$

② $E[Y] = \sum y f_Y(x,y)$

③ $E[XY] = \sum_x \sum_y xy\, f_{X,Y}(x,y)$

Covariance

X and Y be two R.V. then covariance of X and Y is defined as

$$\mathrm{cov}(X, Y) = E\left[X - E[X]\right] E\left[Y - E[Y]\right]$$

$$= E[XY] - E[X]E[Y]$$

Correlation coefficient

The correlation coefficient of two R.V. X and Y is defined as

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

where $\sigma_X > 0$, $\sigma_Y > 0$

Note: $-1 \le \rho(X,Y) \le 1$

Conditional Expectation

(X,Y) be Joint Discrete R.V. then conditional expectation of $g(X,Y)$ given $X=x$ is defined as

$$E[g(x,y)/X=x] = \sum_j g(x_i, y_j)\, f_{Y|X}(y_j|x)$$

i.e. $$E[Y/X=x] = \sum_j y_j\, f_{Y|X}(y_j|x)$$

$$= \sum_j y_j\, P(Y=y_j \mid X=x)$$

Q. For the following bivariate probability distribution of X and Y find

(i) $P(X \le 2, Y = 3)$

(ii) $P(X \le 1)$

(iii) $P(Y = 4)$

(iv) $P(Y \le 5)$

(v) $P(X < 2, Y \le 3)$

X | Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Marginal distribution is given by

X\Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	1/32	2/32	2/32	3/32	8/32
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16
2	1/32	1/32	1/64	1/64	0	2/64	8/64
$p_Y(y)$	3/32	3/32	11/64	13/64	6/32	16/64	1

(1) $P(X \le 2, Y=3) = P(X=0, Y=3) + P(X=1, Y=3) + P(X=2, Y=3)$

$= \frac{1}{32} + \frac{1}{8} + \frac{1}{64}$

$= \frac{11}{64}$

② $P(X \le 1) = P(X=0) + P(X=1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

③ $P(Y=4) = 13/64$

④ $P(Y \le 5) = P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5)$

$1 - P(Y=6)$

$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} + \frac{13}{64} + \frac{6}{32}$

$= \frac{48}{64}$

⑤ $P(X<2, Y \le 3) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=3)$

Q. Let X and Y have joint p.d.f.

Y \ X	−1	0	1
0	b	2b	b
1	3b	2b	b
2	2b	b	2b

Find marginal dist. of ~~X~~ and Y. Also find conditional distribution of X given Y=1.

Q. The joint probability distribution of X and Y is given in the following Table

X\Y	1	3	9
2	$1/8$	$1/24$	$1/12$
4	$1/4$	$1/4$	0
6	$1/8$	$1/24$	$1/12$

1. Find the marginal probability distribution of Y.
2. Find the conditional distribution of Y when $X = 4$
3. Find covariance of X and Y
4. Are X and Y independent?

$X \backslash Y$	1	3	9	$f_X(x)$
2	1/8	1/24	1/12	6/24
4	1/4	1/4	0	2/4
6	1/8	1/24	1/12	6/24
$f_Y(y)$	4/8	8/24	2/12	1

(i) Marginal Probability distribution of Y

$$P(Y=1) = 4/8 = \frac{1}{2}, \quad P(Y=3) = \frac{8}{24} = \frac{1}{3}, \quad P(Y=9) = \frac{2}{12} = \frac{1}{6}$$

(2) conditional dist. of Y when $X = 4$

$$P(Y=1 \mid X=4) = \frac{P(Y=1 \cap X=4)}{P(X=4)}$$

$$= \frac{1/4}{2/4}$$

$$= \frac{1}{2}$$

$P(Y=3/X=4) = 1/2$ $\qquad$ $P(Y=9/X=4) = 0$

(3)

$$E[X] = \sum x f_X(x) = \left(2\times\frac{6}{24}\right)+\left(4\times\frac{2}{4}\right)+\left(6\times\frac{6}{24}\right)$$

$$= 4$$

$$E[Y] = \sum y f_Y(y) = \left(1\times\frac{4}{8}\right)+\left(3\times\frac{8}{24}\right)+\left(9\times\frac{2}{12}\right)$$

$$= 3$$

$cov(X, Y)$

$= E[XY] - E[X]E[Y]$

$$E[XY] = \sum xy f_X(x,y) = \left(2\times\frac{1}{8} + 6\times\frac{1}{24} + 18\times\frac{1}{12}\right) +$$

$$\left(4\times\frac{1}{4} + 12\times\frac{1}{4} + 36\times 0\right) +$$

$$\left(6\times\frac{1}{8} + 18\times\frac{1}{24} + 54\times\frac{1}{12}\right) = 12$$

$$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= 12 - (4)(3)$$

$$= 0$$

X & Y are indp

$P(X=x \cap Y=y)$

$= P(X=x)\,P(Y=y)$

$X=4,\ Y=3$

X & Y not ind.

4

$$P(X=2 \cap Y=1) = \frac{1}{8}$$

Check for all the remaining ones

$$P(X=2) = \frac{6}{24} = \frac{1}{4}$$

$$P(Y=1) = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(X=2)\,P(Y=1) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$

Three fair coins are tossed. X denotes the number of heads on the 1st two coins and Y denotes the number of tails on the last two coins.

(a) Find the joint distribution of X and Y

(b) Find the conditional distribution of Y given $X=1$

(c) Find $\text{cov}(X, Y)$

(d) Find $\rho(X, Y)$

	HHH	HHT	HTH	THH	THT	TTH	HTT	TTT
X :	2	2	1	1	1	0	1	0
Y :	0	1	1	0	1	1	2	2

① Joint distribution of X & Y is

X \ Y	0	1	2	$f_X(x)$
0	0	1/8	1/8	
1	1/8	2/8	1/8	
2	1/8	1/8	0	
$f_Y(y)$				

$X \backslash Y$	0	1	2	$f_X(x)$
0	0	$1/8$	$1/8$	$1/4$
1	$1/8$	$2/8$	$1/8$	$1/2$
2	$1/8$	$1/8$	0	$1/4$
$f_Y(y)$	$1/4$	$1/2$	$1/4$	1

1 Joint Distribution of X & Y

(2) conditional dist. of Y when $x=1$

$$P(Y=0/x=1) = \frac{P(Y=0 \cap X=1)}{P(X=1)}$$

$$= \frac{1/8}{1/2}$$

$$= 1/4$$

$$P(Y=1/x=1) = \frac{P(Y=1 \cap X=1)}{P(X=1)} = \frac{2/8}{1/2}$$

$$= 1/2$$

$$P(Y=2/X=1) = \frac{P(Y=2 \cap X=1)}{P(X=1)}$$

$$= \frac{1/8}{1/2}$$

$$= 1/4$$

$\therefore$ Conditional dist. of Y given $X=1$

$$f_{Y|X}(Y|X=1) = \begin{cases} 1/4 & , \; y=0 \\ 1/2 & , \; Y=1 \\ 1/4 & , \; Y=2 \end{cases}$$

(c)

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

$$E[X] = \sum x f_X(x)$$

$$= \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right)$$

$$= 1$$

$$E[Y] = \sum y f_Y(y)$$

$$= \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right)$$

$$= 1$$

$$E[XY] = \sum xy\, f_{X,Y}(x,y)$$

$$= 0 + \left(1\times\frac{2}{8}\right) + \left(2\times\frac{1}{8}\right) + \left(2\times\frac{1}{8}\right) + (4\times 0)$$

$$= \frac{2}{8} + \frac{2}{4}$$

$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

$\therefore$, $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{3}{4} - 1 = -\frac{1}{4}$

(d) $$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\operatorname{Var}(X) = E[X^2] - (E[X])^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\Rightarrow \sigma_X = \sqrt{\operatorname{Var}(X)} = \frac{1}{\sqrt{2}}$$

$$\operatorname{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\Rightarrow \sigma_Y = \sqrt{\operatorname{Var}(Y)} = \frac{1}{\sqrt{2}}$$

$$\therefore \rho(X,Y) = \frac{-1/4}{1/\sqrt{2} \cdot 1/\sqrt{2}} = \frac{-1/4}{1/2} = -1/2$$

Q. Consider a sample of size 2 drawn without replacement from an urn containing three balls numbered 1, 2 and 3. Assume X is the smaller of the two numbers drawn and Y the larger.

(a) Find the joint discrete density f^n of X & Y

(b) Find the conditional dist. of Y when $X=1$

(c) Find $f(X, Y)$

Joint Continuous Density Function

A two dimensional R.V. (X, Y) is said to be continuous iff $\exists$ a f^n $f_{X,Y}(x,y) \geq 0$ s.t.

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv$$

$f_{X,Y}(x,y)$ or $f(x,y)$ is called Joint Probability density f^n.

Properties

① $f(x,y) \geq 0 \quad \forall \quad (x,y) \in \mathbb{R}^2$

② $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

Note : $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Marginal and conditional Probability Density Function

X, Y Joint Continuous Random Variable

Marginal —

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Conditional —

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad f_X(x) > 0$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Conditional Cumulative Distribution

$$F_{Y/X}(y/x) = \int_{-\infty}^{y} f_{Y/X}(z/x)\,dz$$

$$F_{X/Y}(x/y) = \int_{-\infty}^{x} f_{X/Y}(z/y)\,dz$$

Expectation

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

Joint Moment Generating F^n

The joint moment generating f^n of a two dimensional random variable (X, Y) denoted by $m_{X,Y}(t_1, t_2)$ is defined as

$$m_{X,Y}(t_1, t_2) = E\left[e^{t_1 X + t_2 Y}\right]$$

where t_1 & t_2 are real parameters

Q. Find k so that $f(x,y) = kxy$, $1 \le x \le y \le 2$ will be a joint probability density f^n.

For $f(x,y)$ to be joint prob. density f^n

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$$

$$\Rightarrow \int_{1}^{2}\int_{x}^{2} kxy\, dx\, dy = 1 \quad \rightarrow k\int_{1}^{2} x\left(\int_{x}^{2} y\, dy\right) dx = 1.$$

$$\rightarrow k \int_{1}^{2} x \left[\frac{y^2}{2}\right]_{x}^{2} dx = 1$$

$$k\int_1^2 x\left(\frac{4}{2}-\frac{x^2}{2}\right)dx = 1$$

$$\Rightarrow k\int_1^2 \left(2x - \frac{x^3}{2}\right)dx = 1$$

$$\Rightarrow k\left\{2\left[\frac{x^2}{2}\right]_1^2 - \frac{1}{2}\left[\frac{x^4}{4}\right]_1^2\right\} = 1$$

$$\Rightarrow \frac{9k}{8} = 1$$

$$\Rightarrow k = \frac{8}{9}$$

Q. Find K so that $f(x, y) = K(x+y)$, $0 < x < 1$ and $0 < y < 1$ is a joint probability density f^n.

$\because$ $f(x,y)$ is a joint probability density f^n

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{0}^{1}\int_{0}^{1} K(x+y)\,dx\,dy = 1$$

$$\Rightarrow K \int_{0}^{1} \left\{ (x+y)\,dy \right\} dx = 1 \quad \Rightarrow K\int_{0}^{1} \left[xy + \frac{y^2}{2}\right]_0^1 dx = 1.$$

$$\Rightarrow k\int_0^1 \left(x + \frac{1}{2} - 0\right) dx = 1$$

$$\Rightarrow k\int_0^1 \left(x + \frac{1}{2}\right) dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} + \frac{1}{2}x\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{2} + \frac{1}{2} - 0\right] = 1$$

$$\Rightarrow k = 1$$

Q. Let the joint p.d.f. of X and Y be

$$f(x,y) = \begin{cases} (x+y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P[0 < X < \frac{1}{2},\ 0 < Y < \frac{1}{4}]$

(ii) $E[X]$

(iii) $E[Y]$

(iv) $E[XY]$

(v) $E[X+Y]$

(vi) $\rho[X, Y]$

$$\int\int x f(x,y)\,dx\,dy$$

$$P\left[0<X<\frac{1}{2},\ 0<Y<\frac{1}{4}\right] = \int_0^{1/2}\left[\int_0^{1/4}(x+y)\,dy\right]dx$$

$$= \int_0^{1/2}\left[xy+\frac{y^2}{2}\right]_0^{1/4} dx$$

$$= \int_0^{1/2}\left[\frac{x}{4}+\frac{1}{32}-0\right]dx$$

$$= \int_0^{1/2}\left(\frac{x}{4}+\frac{1}{32}\right)dx$$

$$= \left[\frac{x^2}{8}+\frac{x}{32}\right]_0^{1/2} = \left[\frac{1/4}{8}+\frac{1/2}{32}-0\right] = \frac{3}{64}$$

$$E[X] = \int_0^1 \int_0^1 x f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x(x+y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (x^2 + xy)\, dx\, dy$$

$$= \int_0^1 \left\{ \int_0^1 (x^2 + xy)\, dy \right\} dx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_0^1 dx$$

$$E[X] = \int_0^1 \left(x^2 + \frac{x}{2} - 0\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} - 0$$

$$= \frac{7}{12}$$

$$E[Y] = \int_0^1 \int_0^1 y f(x, y)\, dx\, dy \quad = ?$$

$$E[XY] = \int_0^1 \int_0^1 xy(x+y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (x^2 y + x y^2)\, dx\, dy$$

$$= \int_0^1 \left\{ \int_0^1 (x^2 y + x y^2)\, dy \right\} dx$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{x y^3}{3} \right]_0^1 dx$$

$$E[XY] = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3} - 0 \right) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{6} \right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6} - 0$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

$$E[X+Y] = \int_0^1 \int_0^1 (x+y)(x+y)\,dx$$

$$= \int_0^1 \int_0^1 (x+y)^2\,dx$$

$$= \text{solve this}$$

or

$$E[X+Y] = E[X] + E[Y] = \frac{7}{12} + \frac{7}{12}$$

$$= \frac{7}{6}$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var} X}\,\sqrt{\operatorname{var} Y}}$$

$$\operatorname{cov}(X,Y) = E[XY] - E[X]\,E[Y]$$

$$= \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12}$$

$$= \frac{1}{3} - \frac{49}{144}$$

$$= -\frac{1}{144}$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^1\int_0^1 x^2(x+y)\,dx\,dy = \frac{5}{12}$$

$$E[Y^2] = 5/12$$

Calculate

$$\mathrm{var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$\mathrm{var}(Y) = \frac{11}{144}$$

$$\rho[X,Y] = \frac{-1/144}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = -\frac{1}{11}$$

Q. The probability density f^n of a continuous bivariate distribution is given

$$f(x,y) = x+y, \quad 0 \le x \le 1, \ 0 \le y \le 1$$
$$= 0 \quad , \text{ otherwise.}$$

Find correlation coefficient of x & y.