General Sol^n $\longrightarrow$ Arbitrary constants / Arbitrary fn

$\longrightarrow$ satisfy some other conditions on the boundary which are known as boundary conditions.

BVP ∈ Boundary Value Problem | A partial differential equation together with boundary conditions is known as BVP.

A

Q. Solve the eqn $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ given $u(x,0) = 6e^{-3x}$

Here, u is a fn of x & t

$u = f(x,t)$

let, $u = XT$

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where X is a fn of x only

T " " " t only

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XT) = X'T$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = XT'$$

$$X' = \frac{\partial}{\partial x}(X) = \frac{d}{dx}(X)$$

you may write

$$T' = \frac{\partial}{\partial t}(T) = \frac{d}{dt}(T)$$

Sub. these values in A we get

$$X'T = 2(XT') + XT$$

$$\Rightarrow X'T = X(2T' + T)$$

$$\Rightarrow \frac{X'}{X} = \frac{2T' + T}{T} \quad \text{--- (1)}$$

$$\int \frac{dx}{x} = \log x$$

$\because$ x & t are independent so

as t varies x is regarded as a constant and vice-versa

$\therefore$ LHS & RHS can be considered as constant and consider that constant to be k.

$$\frac{X'}{X} = k \qquad \text{and} \qquad \frac{2T' + T}{T} = k$$

$$\Rightarrow \frac{\frac{dx}{dx}}{X} = k$$

$$\Rightarrow \frac{dX}{X} = k\,dx$$

On integrating we get

$$\int \frac{dx}{x} = \int k\,dx$$

$$\Rightarrow \log x = kx + c_1 \longrightarrow (11)$$

Again,

$$\frac{2T' + T}{T} = k$$

$$\Rightarrow \frac{2T'}{T} + 1 = k \quad \Rightarrow \frac{2T'}{T} = k - 1$$

Solve it

Q. Solve by method of seperation of variables.

(i) $$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad \text{--------} \quad (A)$$

Here, z is a f^n of x & y

Let, $z = XY$ (1) where X is a f^n of x only and Y is a f^n of y only

Now,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(XY) = X'Y \quad , \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = X''Y$$

$$\frac{\partial z}{\partial y} = XY'$$

Sub. these values in (A) we get

$$X''Y - 2X'Y + XY' = 0$$

$$\Rightarrow (X'' - 2X')Y = -XY'$$

$$\Rightarrow \frac{X'' - 2X'}{X} = -\frac{Y'}{Y}$$

$$\Rightarrow \frac{2X' - X''}{X} = \frac{Y'}{Y} \quad \text{--- (ii)}$$

$\because$ X & Y are independent of each other

$\therefore$ Eqn (2) holds iff each side is equal to a constant k (say)

$$\frac{2X' - X''}{X} = k \qquad \text{and} \qquad \frac{Y'}{Y} = k$$

$(D^2 - 2D + k)x = 0$

$$\Rightarrow X'' - 2X' + kX = 0 \qquad\qquad \Rightarrow Y' - kY = 0$$

$(D-k)y = 0$

ODE

A.E.: $\lambda^2 - 2\lambda + k = 0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-4k}}{2} = 1 \pm \sqrt{1-k}$$

$$X = c_1 e^{(1+\sqrt{1-k})x} + c_2 e^{(1-\sqrt{1-k})x}$$

A.E.: $\alpha - k = 0$

$$\Rightarrow \alpha = k$$

$$Y = c_3 e^{ky}$$

$$\therefore\ Z = XY = \left[c_1 e^{(1+\sqrt{1-k})x} + c_2 e^{(1-\sqrt{1-k})x}\right] c_3 e^{ky}$$

(2) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ where $u(x,0) = 4e^{-x}$

(3) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ where $u(0,y) = 8e^{-3y}$

Given, $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ —— (1)

Here, u is a f^n of x & y

let, $u = XY$ —— (11) where X is a f^n of x only
Y is a f^n of y only

Now, $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Sub. the above values in eqn (i) we get

$$X''Y = 4XY'$$

$$\Rightarrow \frac{X''}{X} = 4\frac{Y'}{Y} \quad \text{--- (iii)}$$

$\because$ x & y are independent

$\therefore$ Eqn (iii) holds if each side is equal to a constant k (say)

$$\frac{X'}{X} = k \quad \text{and} \quad \frac{Y'}{Y} = k$$

On integrating $\log X = kx + c_1$ and $\log Y = ky + c_2$

$$X = e^{kx+c_1} \quad \& \quad Y = e^{ky+c_2}$$

$$\Rightarrow X = e^{kx}e^{c_1} \qquad \Rightarrow Y = e^{ky}e^{c_2}$$

$$\Rightarrow X = Ae^{kx} \qquad \Rightarrow Y = Be^{ky}$$

where $A = e^{c_1}$

$B = e^{c_2}$

c_1 & c_2 are integrating const.

$$\therefore \; u = XY = Ae^{kx} \cdot Be^{ky}$$

$$= AB\, e^{k(x+y)}.$$

$$u(x,y) = AB\,e^{k(x+y)} \longleftarrow \text{(IV)}$$

$$u(0,y) = 8e^{-3y}$$

$$\Rightarrow AB\,e^{k(0+y)} = 8e^{-3y}$$

$$\Rightarrow AB\,e^{ky} = 8e^{-3y}$$

comparing both sides we get

$AB = 8$ and $k = -3$

From (IV) we have, $u(x,y) = 8e^{-3(x+y)}$

9. Find a solⁿ of the eqⁿ $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\, g(y)$

Solve the eqⁿ subject to the conditions $u = 0$, $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$ for all values of y.

Given,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{—— (i)}$$

Here, u is a fⁿ of x & y

let, $u = XY$ (ii) where X is a fⁿ of x only and Y is a fⁿ of y only

$$\frac{\partial u}{\partial x} = X'Y \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY'$$

Sub. these values in eqn (1)

$$X''Y = XY' + 2XY$$

$$\Rightarrow X''Y = X(Y' + 2Y)$$

$$\Rightarrow \frac{X''}{X} = \frac{Y' + 2Y}{Y} \quad \text{--- (iii)}$$

$\because$ x & y are independent

$\therefore$ Eqn (iii) holds only if both sides is equal to a constant k (say)

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{Y' + 2Y}{Y} = k$$

$\Rightarrow X'' - kX = 0$

A.E. : $\lambda^2 - k = 0$

$\Rightarrow \lambda = \pm\sqrt{k}$

$X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

$Y' + 2Y = kY$

$\Rightarrow Y' = (k-2)Y \Rightarrow \frac{Y'}{Y} = k-2$

On integrating we get

$\log Y = (k-2)y + c_3$

$\Rightarrow Y = Ae^{(k-2)y}$

where $A = e^{c_3}$

$\therefore \quad u = XY = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right)\left(Ae^{(k-2)y}\right)$ — IV

Now,

$$u = 0 \quad , \quad \frac{\partial u}{\partial x} = 1 + e^{-3y} \qquad \text{when} \quad x = 0$$

$$\therefore \quad u(0, y) = 0$$

$$(c_1 + c_2)\left(Ae^{(k-2)y}\right) = 0$$

$$\Rightarrow c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

Using the above value in (iv) we get

$$u(x, y) = \left(c_1 e^{\sqrt{k}x} - c_1 e^{-\sqrt{k}x}\right)\left(Ae^{(k-2)y}\right)$$

$$= c_1 A \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) e^{(k-2)y} \quad \text{—— (v)}$$

$$\frac{\partial u}{\partial x} = C_1 A \left(\sqrt{k}\, e^{\sqrt{k}x} + \sqrt{k}\, e^{-\sqrt{k}x}\right) e^{(k-2)y}$$

$$\frac{\partial}{\partial x} e^{\sqrt{k}x}$$

$$= e^{\sqrt{k}x}\left(\frac{\partial}{\partial x}\sqrt{k}x\right)$$

$$= \sqrt{k}\, e^{\sqrt{k}x}$$

$$\frac{\partial}{\partial x} e^{-\sqrt{k}x}$$

$$= -\sqrt{k}\, e^{-\sqrt{k}x}$$

At $x=0$, $\quad \frac{\partial u}{\partial x} = 1 + e^{-3y}$

$$\Rightarrow C_1 A \left(\sqrt{k} + \sqrt{k}\right) e^{(k-2)y} = 1 + e^{-3y}$$

$$\Rightarrow 2 C_1 A \sqrt{k}\, e^{(k-2)y} = 1 + e^{-3y}$$

consider, $\quad 2 C_1 A \sqrt{k}\, e^{(k-2)y} = 1$

$$\Rightarrow 2 C_1 A \sqrt{k}\, e^{(k-2)y} = e^{0y}$$

$\Rightarrow \quad 2C_1 A\sqrt{k} = 1$

and $\quad k - 2 = 0 \quad \Rightarrow k = 2$

then $\quad 2C_1 A\sqrt{2} = 1$

$\Rightarrow \quad C_1 A = \dfrac{1}{2\sqrt{2}}$

From (v)

$$u_1 = \frac{1}{2\sqrt{2}} \left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) e^{0y}$$

$$= \frac{1}{2\sqrt{2}} \left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) = \frac{\sinh\sqrt{2}x}{\sqrt{2}} \quad \text{—— (vi)}$$

Again,

let, $2c_1 A\sqrt{k}\, e^{(k-2)y} = e^{-3y}$

$\Rightarrow k = -1$

$2c_1 A\sqrt{-1} = 1$

$\Rightarrow c_1 A = \dfrac{1}{2i}$

$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

From (v)

$u_2 = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) e^{-3y} = \sin x\, e^{-3y}$

$\therefore u = u_1 + u_2 = \dfrac{\sinh\sqrt{2}x}{\sqrt{2}} + \sin x\, e^{-3y}$