One Dimensional Wave eq^n (vibration of a stretched string)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions which the eq^n $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are

(i) $y = 0$ when $x = 0$

(ii) $y = 0$ when $x = l$

} This must be true for every value of t

l = length

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the initial conditions are

(i) $y = f(x)$ when $t = 0$

(ii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$

Soln of the Wave Eqn

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Sol^n of the wave eq^n satisfying boundary condit^n is

$$y = \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad ; \quad n = 1, 2, 3, \ldots$$

(*)

or $$y = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$ is also a sol^n

(*)

Also if $y(x,0)$ is known then

$$y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

(*)

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Q. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

Soln: The boundary conditions are

$$y(0,t)=0 \quad , \quad y(l,t)=0$$

Here,
$f(x) = y(x,0)$

The initial conditions are

$$y(x,0) = a \sin\frac{\pi x}{l}$$

$$\frac{\partial y}{\partial t} = 0 \quad \text{when} \quad t = 0$$

Soln is given by

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \longrightarrow (*)$$

$$\text{where} \quad a_n = \frac{2}{l}\int_0^l y(x,0) \sin\frac{n\pi x}{l}\, dx$$

$$a_n = \frac{2}{l}\int_0^l a \sin\frac{\pi x}{l} \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2a}{l}\int_0^l \sin\frac{\pi x}{l} \sin\frac{n\pi x}{l}\,dx$$ which vanishes for all values other than $n=1$

Now,

$$a_1 = \frac{2a}{l}\int_0^l \sin\frac{\pi x}{l} \sin\frac{\pi x}{l}\,dx$$

$$= \frac{2a}{l}\int_0^l \sin^2\frac{\pi x}{l}\,dx$$

$$= \frac{a}{l}\int_0^l \left(1 - \cos\frac{2\pi x}{l}\right)dx$$

$$a_1 = \frac{a}{l}\left[x - \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l$$

$$= a$$

for $n = 1$, we get from eq ✱

$$y(x,t) = a\cos\frac{\pi c t}{l}\sin\frac{\pi x}{l}$$

Solution of Wave Eq^n

The wave eq^n is given as

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

let $\quad y = XT \quad$ --- (11)

where X is a f^n of x only and T is a f^n of t only.

Then,
$$\frac{\partial^2 y}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Sub. the above values in ① we get

$$X T'' = c^2 X'' T$$

$$\Rightarrow \frac{T''}{T} = c^2 \frac{X''}{X}$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} \quad \text{--- (iii)}$$

$\because$ LHS is a f^n of x only and RHS is a f^n of t only and x & t are independent of each other

$\therefore$ For both sides to be equal, they must be equal to some constant say k.

From (iii) we have

$$X'' - kX = 0 \quad \text{and} \quad T'' - kc^2T = 0 \qquad \text{(iv)}$$

Case I : $k > 0$ i.e. positive, let $k = p^2$ then

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$T = c_3 e^{cpt} + c_4 e^{-cpt}$$

show the soln process

Case II : $k < 0$ i.e. negative, let $k = -p^2$ then

$$X = c_1 \cos px + c_2 \sin px$$

$$T = c_3 \cos cpt + c_4 \sin cpt$$

Case III : $K = 0$ then

$$X = c_1 x + c_2$$

$$T = c_3 t + c_4$$

The various possible sol^{n}s of eq^n (1) are

$$y = XT$$

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$y = (c_1 x + c_2)(c_3 t + c_4)$$

$\because$ we are dealing with a problem on vibrations

$\therefore$ y must be periodic f^n of x and t.

$\Rightarrow$ The sol^n must contain trigonometric f^n

$$\Rightarrow y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{—} \ (A)$$

is the only suitable sol^n for the wave eq^n

and it corresponds to $k = -p^2$.

Applying the boundary conditions

$y = 0$ when $x = 0$

and $y = 0$ when $x = l$

$$\Rightarrow \quad 0 = c_1 (c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (v)}$$

and $$0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (vi)}$$

From (v) we have $c_1 = 0$

then (vi) $\Rightarrow$ $c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$

$\sin pl = 0 \Rightarrow pl = n\pi \rightarrow p = \frac{n\pi}{l}$ where $n = 1, 2, 3, \ldots$

$\therefore$ A solⁿ of the wave eqⁿ satisfying the boundary conditⁿ is that

$$y = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right)$$

$$= \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right)$$ where $a_n = c_2 c_3$, $b_n = c_2 c_4$

Adding up solⁿ for various values of n we have

$$y = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \quad (2\text{B})$$ is also a solⁿ

Applying initial condit^n we get

$$y = f(x) \quad \text{when} \quad t = 0$$

$$\frac{\partial y}{\partial t} = 0 \quad \text{when} \quad t = 0$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \qquad \text{— vii}$$

$$\text{and} \quad 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \, b_n \sin \frac{n\pi x}{l} \qquad \text{— viii}$$

Eqn vii represents a Fourier series

$$\therefore \quad a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

From eqn (viii) $b_n = 0$

$$\therefore \quad y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \qquad (\text{using } (B))$$

when $y(x,0)$ is known

Q. A tightly stretched string with fixed end points $x=0$ and $x=l$, is initially in a position given $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y(x,t)$.

Soln: Wave Eqn

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (I)}$$

Boundary conditions,

$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right] \text{—— (II)}$$

Initial conditions,

$$\left.\begin{aligned} y(x,0) &= y_0 \sin^3\left(\frac{\pi x}{l}\right) \\ \left(\frac{\partial y}{\partial t}\right)_{t=0} &= 0 \end{aligned}\right] \text{—— (III)}$$

Solution is given by

$$y = y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \quad , \quad 0 < x < l \quad \text{—(iv)}$$

where $$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

and $$y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) = f(x)$$

Now,

$$a_n = \frac{2}{l}\int_0^l y_0 \sin^3\left(\frac{\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2y_0}{l}\int_0^l \left(\frac{3\sin\left(\frac{\pi x}{l}\right) - \sin\frac{3\pi x}{l}}{4}\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{y_0}{2l}\int_0^l \left(3\sin\frac{\pi x}{l}\sin\frac{n\pi x}{l} - \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\right)dx \quad \text{—(v)}$$

$$= \frac{3y_0}{4l}\int_0^l 2\sin\frac{n\pi x}{l}\sin\frac{\pi x}{l}\,dx - \frac{y_0}{4l}\int_0^l 2\sin\frac{n\pi x}{l}\sin\frac{3\pi x}{l}\,dx$$

$$\Rightarrow a_n = \frac{3y_0}{4l}\int_0^l\left[\cos(n-1)\frac{\pi x}{l} - \cos(n+1)\frac{\pi x}{l}\right]dx - \frac{y_0}{4l}\int_0^l\left[\cos(n-3)\frac{\pi x}{l} - \cos(n+3)\frac{\pi x}{l}\right]dx$$

$(\because 2\sin A\sin B = \cos(A-B) - \cos(A+B))$

$$a_n = \frac{3y_0}{l}\left[\frac{\sin(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}} - \frac{\sin(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}}\right]_0^l - \frac{y_0}{4l}\left[\frac{\sin(n-3)\frac{\pi x}{l}}{(n-3)\frac{\pi}{l}} - \frac{\sin(n+3)\frac{\pi x}{l}}{(n+3)\frac{\pi}{l}}\right]_0^l$$

$\Rightarrow a_n = 0 \ \forall\, n$ except $n=1$, $n=3$. Also n varies from 1 to ∞.

From (v),

$$a_1 = \frac{y_0}{2l}\int_0^l \left(3\sin\frac{\pi x}{l}\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\sin\frac{\pi x}{l}\right)dx$$

$$= \frac{3y_0}{2l}\int_0^l \sin^2\frac{\pi x}{l}\,dx - \frac{y_0}{2l}\cdot\frac{1}{2}\int_0^l 2\sin\frac{3\pi x}{l}\sin\frac{\pi x}{l}\,dx$$

$$= \frac{3y_0}{2l}\int_0^l \frac{\left(1-\cos\frac{2\pi x}{l}\right)}{2}dx - \frac{y_0}{4l}\int_0^l\left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}\right)dx$$

$(\because 2\sin A\sin B = \cos(A-B) - \cos(A+B))$

$$= \frac{3y_0}{4l}\left[x - \frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right]_0^l - \frac{y_0}{4l}\left[\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}} - \frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right]_0^l$$

$$= \frac{3y_0}{4l}(l) \qquad (\because \sin n\pi = 0 \ \forall\, n\in\mathbb{Z})$$

$$= \frac{3y_0}{4}$$

Again,

from (v)

$$a_3 = \frac{y_0}{2l}\int_0^l \left(3\sin\frac{\pi x}{l}\sin\frac{3\pi x}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi x}{l}\right)dx$$

$$= \frac{3y_0}{4l}\int_0^l 2\sin\frac{3\pi x}{l}\sin\frac{\pi x}{l}\,dx - \frac{y_0}{2l}\int_0^l \sin^2\frac{3\pi x}{l}\,dx$$

$$= \frac{3y_0}{4l}\int_0^l \left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}\right)dx - \frac{y_0}{4l}\int_0^l\left(1-\cos\frac{6\pi x}{l}\right)dx$$

$$= \frac{3y_0}{4l}\left[\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}} - \frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right]_0^l - \frac{y_0}{4l}\left(x - \frac{\sin\frac{6\pi x}{l}}{\frac{6\pi}{l}}\right)_0^l$$

$$= \frac{-y_0}{4} \qquad (\because \sin n\pi = 0 \quad \forall\, n \in \mathbb{Z})$$

$$\therefore\; y = y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$= a_1\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} + a_3\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l} + 0 + 0 + \dots$$

$$= \frac{3y_0}{4}\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \frac{y_0}{4}\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}$$

$\therefore\ y = \frac{y_0}{4}\left[3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right]$

Q. A string is stretched between the fixed points (0,0) and (l,0) and released from the initial deflection given by

$$f(x) = \begin{cases} \frac{2k}{l}x, & 0 < x < l/2 \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

Find the deflection of string at any time t.

Soln: Wave Equation is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

Boundary conditions,

$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right] \quad \text{——— (ii)}$$

Initial conditions,

$$\left.\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$$y(x,0) = \begin{cases} \frac{2k}{l}x \,, & 0<x<l/2 \\ \frac{2k}{l}(l-x) \,, & \frac{l}{2}<x<l \end{cases} \quad \text{— (iii)}$$

Soln is given by

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \,, \quad 0<x<l \quad \text{— (iv)}$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$, $y(x,0) = f(x)$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2k}{l}x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$\Rightarrow a_n = \frac{4k}{l^2}\left[x\left(\frac{-\cos\frac{k\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \frac{4k}{l^2}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}$$

$$= \frac{4k}{l^2}\left[\left(\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + \left(0+0+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$ ($n \neq 0$ as n varies 1 to ∞)

$$\therefore\ y(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$